Foundation Example of Invalidity of the Upper Bound Limit Theorem in Plastic Theory in Case of Non-fixed Boundary

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SYNOPSIS
An infinitely rigid strip footing on an ideal plastic pure cohesion soil with associated flow rule is considered. The footing is loaded eccentrically with a load that has both a vertical and a horizontal component orthogonal to the strip footing.

Among the kinematically admissible failure mechanisms there are mechanisms by which the footing is lifted off the soil surface along the one edge of the footing. It is shown that there is such a mechanism that gives a lower carrying capacity than that obtained by a statically admissible stress field. Thus the upper bound theorem of the plastic theory is violated. The standard indirect proof of the upper bound theorem is examined in order to see why it may fail in case of a non-fixed boundary between soil and structure.

A solution to the carrying capacity problem is obtained by introducing an effective footing width together with an assumption of contact between soil and footing under preservation of the associated flow rule also where lift-off takes place. The effective footing width is defined such that there just accurately is no lift-off for the exact plastic solution obtained for the width reduced footing.

1 Introduction
In connection with the reliability analysis of the anchor blocks for the large suspension bridge across the Eastern Channel of the Great Belt in Denmark it is relevant to consider an undrained failure mode in the soil where the anchor block rotates such that there is lift-off from the soil surface of the rear end of the anchor block.

Undrained failure in water saturated soil under compression can be modeled in terms of the ideal plasticity theory for pure cohesion soil. This is also the case even under tension if cracks or voids cannot be formed inside the material or loss of contact with bodies at the boundaries (lift-off) cannot occur. For simplicity an infinitely long strip footing is considered in the following. It is subjected to an eccentric vertical line load and a horizontal line load orthogonal to the strip and directed towards what will be denoted as the front edge of the footing. The opposite edge is the rear edge.

The usual tools of plastic carrying capacity analysis are the static theorem (the lower bound theorem) and the kinematic theorem (the upper bound theorem). These theorems are fundamentally dependent on the validity of the associated flow rule (the normality condition for the strain rate vector relative to the yield condition). If the
excentricity of the vertical load is sufficiently large the critical failure figure may show lift-off of the strip footing along the rear edge of the footing. However, in this situation the lower and upper bound theorems are not valid. Their indirect proofs are restated in Appendix 1 and it is clarified at which point these proofs fail.

2 Plastic analysis in case of lift-off

Lift-off can be considered as a violation of the associated flow rule. The yield condition is shown in a $(\sigma, \tau)$-diagram $(\sigma = \text{normal compression stress}, \tau = \text{shear stress})$ on top of the corresponding $(\varepsilon, \gamma)$-diagram $(\varepsilon = \text{normal strain}, \gamma = \text{shear strain})$ in Figure 1.

![Diagram](image)

Figure 1. Yield conditions for cohesion soil with zero tensile strength. Inside the soil body the model a) is applicable. At boundaries with separation (cracks, lift-offs) the model b) should be applied.

If the associated flow rule is required to be valid the consequence is that there is contact between the footing surface and the soil surface also during lift-off and this contact contributes to the resistance against plastic collapse. The lift-off line may be modeled as a rupture line (discontinuity line of the velocity field) along the footing surface starting at the rear end. The corresponding plastic dissipation is derived in Appendix 2. Such a resistance part may be considered to be unrealistic and to the unsafe side, of course. However, admitting that no stresses can be transferred from the footing to the soil along the lift-off line simultaneously with assuming full contact violates the associated flow rule. This makes the lower and upper bound theorems invalid. If the assumption of contact is abandoned it may be claimed that the associated flow rule for the plastic material is still valid but in this case the boundary becomes non-fixed and as shown in Appendix 1, for such boundaries the proofs of the lower and upper bound theorems break down.

It will be demonstrated in the next section that if the upper bound theorem is used searching through the set of kinematically admissible failure figures that allow a non-fixed boundary of the type of stress free lift-off along the rear edge of the footing then the resulting smallest upper bound of the carrying capacity is smaller than a lower bound obtained from a statically admissible stress distribution in the soil.

A way to overcome this contradiction is to maintain the associated flow rule also along the lift-off part, that is, to introduce an artificial plastic dissipation along the line of lift-off as computed in Appendix 2. Searching through the set of all kinematically admissible failure figures will then give a lowest upper bound that coincides with the largest lower bound obtained by searching through all statically admissible stress distributions. If the failure figure corresponding to the lowest upper bound shows lift-off along the rear edge the solution is unrealistic. A step towards more realism of the modeling is then to let the obtained
lift-off part of the footing be inactive, that is, to fictively cut off that part of the footing. A new search through the set of kinematically admissible failure figures for this smaller footing maintaining dissipation at the lift-off part leads to a new critical failure figure. If this also has lift-off, the procedure is repeated. If the solution shows tangency between the rupture circle and the footing surface at the rear end of the reduced footing it is a solution to which there is a corresponding statically admissible stress field. If there is a non-zero angle between the rupture circle and footing surface it indicates that too much of the footing has been removed. Then a piece should be added and the computation repeated.

For homogeneous soil the solution is given analytically in the next section. It is noted that even though the unrealistic assumption of having dissipation in the lift-off line is maintained for the fictively reduced footings the solution itself is made realistic since it only depends on maintaining the assumption between the point of tangency and the front end of the footing. During proportional loading the lift-off corresponding to the solution will be reached before further lift-off makes the plastic limit analysis invalid.

3 Counter-example to the limit theorems

The bearing capacity of the eccentrically loaded strip footing, placed on the surface of a homogeneous undrained clay with shear strength \( c_u \), may be investigated by the rupture figure BCDE shown in Figure 2.

The footing is assumed to be loaded at failure by the vertical line load \( V \), acting at the distance \( a \) from the edge \( P \) of the footing, and the horizontal line load \( H \). During failure the footing is assumed to lift off from the soil surface to the left of the boundary point \( B \). The width \( b \) of the part BP which remains in contact with the soil is assumed to be smaller than the total width of the footing.

![Figure 2. Eccentrically loaded strip footing.](image)

The rupture figure consists of a rigid body of soil BCP which is bounded by the line rupture BC formed as a circle arc with centre \( O \) and radius \( R \), the radial zone PCD, and the Rankine zone PDE. The geometry of the rupture figure is seen to be completely determined by \( R \) and the centre angle \( \theta \) for the circle arc BC. Thus, the width BP is given by:

\[
b = R \tan \theta
\]

and the radius PC of the radial zone by:

\[
r = R \left( \frac{1}{\cos \theta} - 1 \right)
\]

The rupture figure defines a statically admissible solution to the bearing capacity problem, the state of stress being known everywhere in the rupture zones PCDE and along the line rupture BC. The equilibrium for the rigid body BCP can be shown to require:

\[
V = c_u R \left( \frac{\pi}{2} + 1 + 2\theta \right) \tan \theta
\]

\[
H = c_u R (2\theta - \tan \theta)
\]

and

\[
V a = c_u R^2 \left[ \left( \frac{\pi}{4} + \frac{1}{2} + \theta \right) \tan^2 \theta - \tan \theta + \theta \right]
\]

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\( \theta \) may be found from the first two of these equations when the ratio \( H/V \) is given. \( R \) may then be found from the first and third equation when also \( a \) is known. Any one of the equations will then determine either the value of \( c_n \) which is required to take a given load, or the load which produces failure for a given value of \( c_n \).

The rupture figure can also be taken to define a kinematically admissible solution, the displacement field at failure being defined by a rotation (clockwise) of the footing and the rigid body BCP about the centre \( O \). Putting the work rate \( W_e \) of the external load equal to the plastic dissipation \( W_d \) in the rupture figure, one gets for a unit rotation about \( O \):

\[
V(R \tan \theta - a) + HR = c_n R^2 \left[ \theta + \left( \frac{\pi}{4} + \frac{1}{2} + \theta \right) \tan^2 \theta \right]
\]

It is easy to show that the results from the statically admissible solution satisfy this equation.

The work equation may be used to find the value of \( c_n \) which corresponds to failure for given values of \( V \) and \( H \) and any chosen set of parameters \( R \) and \( \theta \). If the limit theorems hold, these parameters should be chosen so that \( c_n \) as found by the work equation attains a maximum value. This extremum condition may be satisfied by putting \( \partial W_e / \partial R = 0 \) and \( \partial W_e / \partial \theta = 0 \). In this way one obtains:

\[
V = c_n R \left[ 1 + \left( \frac{\pi}{2} + 1 + 2\theta \right) \tan \theta \right]
\]

\[
H = c_n R (2\theta - \tan \theta)
\]

and, inserting this in the work equation itself:

\[
V a = c_n R^2 \left[ \left( \frac{\pi}{4} + \frac{1}{2} + \theta \right) \tan^2 \theta + \theta \right]
\]

This is different than the result obtained by the equilibrium equations. For the same values of \( H/V \) and \( a \) the kinematically admissible solution gives a smaller rupture figure, as sketched with dotted lines on the figure, and also a higher value of \( c_n \). For the example shown in the figure, the difference is about 5%.

The difference between the two solutions might be described by saying that the kinematically admissible solution would also be statically admissible if an additional vertical load \( \Delta V = c_n R' \) were acting on the footing at the point \( B' \). However, a more useful point of view is obtained in the following way.

If one uses as parameters in the kinematically admissible solution the coordinates \( x, y \) to the centre \( O \), then the same result as in the statically admissible solution is obtained when the extremum condition is taken as:

\[
\frac{\partial W_e}{\partial x} = \frac{\partial W_d}{\partial x} - c_n R
\]

\[
\frac{\partial W_e}{\partial y} = \frac{\partial W_d}{\partial y}
\]

The last term in the first equation reflects the influence of the moving boundary condition at the point \( B \).

The correction term \(-c_n R\) is consistent with the more generally applicable iteration procedure suggested in the previous section. In fact, according to Appendix 2 the dissipation contribution from the lift-off line \( BB' \) is

\[
W_e(BB') = c_n \int_0^{b-x} \left( \sqrt{z^2 + R^2} - z \right) dz
\]

where \( z = \xi - x \). Then

\[
\frac{\partial}{\partial x} W_e(BB') = c_n \left[ R - 2y \sqrt{(b - x)^2 + R^2} \right]
\]

\[
= -c_n R
\]

for \( x \to b \)
APPENDIX 1

Investigation of the indirect proofs of the lower and upper bound theorems in the plasticity theory

Consider a proportionally loaded structure made of an elastic-ideal plastic material obeying the associated flow rule (the normality condition). If an equilibrium stress distribution in the structure is such that the stress tensor is everywhere strictly inside the yield condition (that is, inside the yield condition surface in the stress tensor space), the stress distribution is said to be safe. A stress distribution is said to be admissible if the stress tensor at no point of the structure is outside the yield condition.

The first part of the lower bound theorem has the formulation:

*A structure can carry the external load if there exists a safe stress distribution in the structure in equilibrium with the external load.*

The easiest way of proof is indirect. By proportional loading the load parameter cannot be larger than the value that corresponds to collapse. Assume that there is a safe equilibrium stress distribution \( Q^* \) that corresponds to the collapse load. Corresponding to the same load there is an equilibrium stress distribution \( Q \) with the stress tensor situated on the yield condition at all points in the structure at which there are non-zero plastic strain rates. The difference \( Q - Q^* \) is an eigenstress distribution. Therefore the virtual work is zero for any kinematically admissible strain rate field. In particular this is so for the plastic strain rate field corresponding to \( Q \). However, this is in conflict with the associated flow rule and the convexity of the yield condition. Thus the load parameter corresponding to the safe stress distribution must be less than the collapse load parameter.

The first part of the upper bound theorem can be given the following formulation:

*A structure cannot carry the external load if there is a kinematically admissible displacement velocity field by which the work rate \( W_e \) of the external load is larger than the total plastic dissipation \( W_d \).*

A displacement velocity field is said to be kinematically admissible if it is consistent both with the geometrical boundary conditions and the associated flow rule. The indirect proof is as follows. Assume that the theorem is wrong, that is, assume that \( W_e > W_d \) and that there is an admissible stress field in equilibrium with the external load. By the principle of virtual work we then get \( W_e = W_i \) where \( W_i \) is the internal work of the statically admissible stress field by the kinematically admissible velocity field. Thus it follows that \( W_d - W_i < 0 \).

This is in conflict with the fact that the convexity of the yield condition and the associated flow rule imply that \( W_d - W_i \geq 0 \).

It is a decisive hidden assumption in these indirect proofs that the safe or the admissible stress field, respectively, does not act on those parts of the boundary that are separated into two boundaries. Such separation may take place if cracks are introduced in the material or it may take place at the interface between a rigid body and the material. An example is a footing where lift-off from the soil surface can occur under the given kinematically admissible velocity field. If the safe or the admissible stress field, respectively, acts on such a separation boundary the equation \( W_e = W_i \) is modified to \( W_e + W_e' = W_i \) in which \( W_e' \) is the work rate from that part of the stress field that under the velocity field acts as an external stress field on the structure. In the last proof the assumption \( W_e > W_d \) then gives \( W_d - W_i + W_e' < 0 \). Since \( W_e' \) may be negative this inequality is not necessarily in conflict with the fact that \( W_d - W_i \geq 0 \).

That \( W_e' \) can be negative is seen in the
strip footing example. The rupture figure shown with dotted line in Figure 2 corresponds to a kinematically admissible velocity field that can be used as a virtual velocity field applied to the stress field that corresponds to the exact carrying capacity solution corresponding to the rupture figure drawn in full line. This gives an external work rate contribution $W'_{e}$ from the stresses acting within $BB'$ on the footing block. With a horizontal velocity component at $B$ equal to $R$ we obviously get $W'_{e} = -c_{u}hR$ asymptotically for $h \to 0$ where $h$ is the distance from $B$ to $B'$. In fact we found in the footing example that the carrying capacity obtained by equating $W_{e}$ and $W_{d}$ for the failure figure with tangent point at $B'$ is less than the exact carrying capacity.

The second part of the lower bound theorem is obvious. It reads

\begin{equation}
A \text{ structure cannot carry an external load for which equilibrium cannot be obtained for any admissible stress distribution in the structure.}
\end{equation}

The second part of the upper bound theorem is as follows:

\begin{equation}
A \text{ structure can carry the external load if all kinematically admissible displacement velocity fields have the property that the external work rate $W_{e}$ is smaller than the corresponding total plastic dissipation $W_{d}$.}
\end{equation}

Again an indirect proof is easy. Consider a strain rate field which corresponds to collapse at a proportionally reduced load and let $W_{e}$ be the external work rate for this strain rate field. Then the work rate for the collapse load is $\lambda W_{e}$ for some $\lambda \leq 1$. Since $\lambda W_{e} = W_{d}$ we have $W_{d} \leq W_{e}$ which is in conflict with the assumption that $W_{d} > W_{e}$.

This proof is seen to be valid also in case of imposed cracks or lift-off since these particular boundary parts are not subject to stresses at the collapse load.

\section*{APPENDIX 2}

\subsection*{Dissipation in lift-off line}

The velocity field $(v_{1},v_{2})$ shown in Figure 3 is within the layer $-\delta \leq y \leq 0$ defined by

\begin{equation}
v_{1} = \omega R y/\delta, \quad v_{2} = \omega R y \tan \theta/\delta
\end{equation}

The velocity is zero at the layer boundary $y = 0$. At the boundary $y = -\delta$ the velocity field coincides with the velocity field obtained by rotating the boundary as a rigid body with angular velocity $\omega$ about a point in distance $R$ below the boundary. The angle $\theta$ is the angle between the y-axis and line from the centre of rotation and the actual point $(x,-\delta)$ on the boundary.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3}
\caption{Velocity field for discontinuity line.}
\end{figure}

Noting that $d\tan \theta/dx = 1/R$, the strain rate tensor is (here letting elongation strains as well as tensile stresses be positive)

\begin{equation}
= \frac{\omega}{2\delta} \begin{bmatrix}
\frac{1}{2}(v_{1,2} + v_{2,1}) \\
\frac{1}{2}(v_{1,2} + v_{2,1}) & v_{2,2}
\end{bmatrix}
\end{equation}
where \( z = x - x_0 \). The two eigenvalues are the principal strain rates
\[
\begin{aligned}
\epsilon_1 & = \frac{\omega}{2\delta} \left[ z \pm \sqrt{z^2 + (R + y)^2} \right] \\
\epsilon_2 & = \frac{\omega}{2\delta} \left[ z \mp \sqrt{z^2 + (R + y)^2} \right]
\end{aligned}
\]  
(16)

According to the associated flow rule these principal plastic strain rates correspond to zero shear stress and zero normal stress for the positive principal strain rate \( \epsilon_1 \) and the normal stress \(-2c\) for the negative principal strain rate \( \epsilon_2 \). Per unit length at \( x \) of the layer \(-\delta \leq y \leq 0\) the dissipation then is
\[
W_d(x; \delta) = -2 \int_{-\delta}^{0} c(x, y) \epsilon_2(x, y) dy
\]
\[
= \frac{\omega}{\delta} \int_{-\delta}^{0} c(x, y) \sqrt{z^2 + (R + y)^2} - z \right) dy
\]
(17)

which for a continuous cohesion function \( c(x, y) \) in the limit \( \delta \rightarrow 0 \) becomes
\[
W_d(x) = \omega c(x, 0) \left( \sqrt{z^2 + R^2} - z \right)
\]  
(18)

The total dissipation over the length from \( x_0 \) to \( x_0 + a \) then becomes
\[
W_d = \omega \int_{0}^{a} c(x_0 + z, 0) \left( \sqrt{z^2 + R^2} - z \right) dz
\]
(19)

which in case the cohesion is a constant \( c \) becomes
\[
W_d = \frac{1}{2} \omega ca^2 \sqrt{(\frac{R}{a})^2 + 1}
\]
\[
\frac{(R}{a})^2 \log \left\{ \left[ 1 + \sqrt{(\frac{R}{a})^2 + 1} / \frac{R}{a} \right] - 1 \right\}
\]  
(20)

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Derivation of the equilibrium equations (3) to (5) in the paper

The shear stress along the rupture line BCDE in the rupture figure in Figure 2 is the cohesion shear stress $c (= c_u)$. The normal stresses are obtained by use of Kötter’s equilibrium equation for the stresses in a rupture line in pure weightless cohesion soil.

Kötter’s equation implies that the normal stress to a circular rupture line on top of a constant normal stress varies along the circle in the opposite direction of the shear stress $c$ proportional to the clockwise tangent rotation with proportionality constant $2c$. Thus the normal stress is constant along a straight rupture line.

The statically correct angle between a rupture line and a stress free surface is $\pi/4$ (as in the right end point E of the considered rupture figure), and the normal stress in this point is $c$. Thus the normal stress on the straight rupture line from D to E is equal to $c$. Consequently the normal stress decreases linearly with arch length from $c(1+\theta+\pi/2)$ at C to $c$ at D on the circle from C to D. Finally it follows that the normal stress on the circle from B to C increases linearly with arch length from $c(1+\pi/2)$ at B to $c(1+\theta+\pi/2)$ at C.

By taking moment about P of the external stresses on the part of the rupture figure situated to the right of the straight line from C to P it follows hereafter by equilibrium that the constant normal stress on the straight rupture line from C to P is $c(2\theta+1+\pi/2)$. Finally the equations (3) to (5) in the paper follows by equilibrium of the rigid body BCP.

The special case $b = 2a$ gives the limit $\theta = 0$, $R\theta = R\tan\theta = b$, $H = cb$, $V = cb(1+\pi/2)$. It is noted that the presence of the horizontal load $H = cb$ makes the vertical carrying capacity $V$ half the known solution $V = cb(2+\pi)$ valid for $H = 0$. This solution cannot be obtained as a limit result from the considered rupture figure. The relevant rupture figure consists of a radial zone as considered here but with $\theta = \pi/4$ and a triangular rigid body on each side of the radial zone.
Derivation of the plastic dissipation in the rupture figure

The plastic dissipation corresponds to clockwise rotation about point O. In the rupture circle BC it is directly obtained as \(cR^2\theta\). In the circle CD it is \(cRr(\theta + \pi/4)\) and in the straight line DE it is \(cRr\). For the outer surface of the rupture figure this sums up to

\[
cR^2[\theta + (\theta + 1 + \pi/4)r/R]
\]

(1)

In the radial rupture zone the total radial shear strain rate is 1 (absolute value of gradient along radius vector of the given velocity field orthogonal to the radius vector) and the total tangential shear strain rate is \((R + r - \rho)/\rho\) at the polar position \((\rho, \alpha)\) with pole at P. The corresponding normal strain rates are zero everywhere. Thus the plastic dissipation in the radial zone is

\[
c \int_{\theta}^{\pi/4} \int_{0}^{r} \left(1 + \frac{R + r - \rho}{\rho}\right) \rho \, d\rho \, d\alpha = c(\theta + \frac{\pi}{4})(R + r)r
\]

(2)

In the Rankine zone PDE the total shear strain rate is 1 (derived as in the radial zone) implying that the dissipation becomes \(c\) times the area of the zone, that is, \(cr^2/2\).

Consequently the total plastic dissipation is

\[
W_d = cR^2 \left[\theta + \left(\theta + 1 + \frac{\pi}{4}\right)\frac{r}{R} + \left(1 + \frac{r}{R}\right)\frac{r}{R}\left(\theta + \frac{\pi}{4}\right) + \frac{1}{2}\left(\frac{r}{R}\right)^2\right]
\]

\[
= cR^2 \left[\theta + \frac{r}{R}\left(2 + \frac{r}{R}\right)\left(\theta + \frac{1}{2} + \frac{\pi}{4}\right)\right] = cR^2 \left[\theta + \left(\theta + \frac{1}{2} + \frac{\pi}{4}\right)\tan^2\theta\right]
\]

(3)

Thus Eq. (6) in the paper follows from the balance \(W_e = W_d\).

Remark on the interpretation of the lower and upper bound theorems in relation to rupture figures

It is obvious that a given safe stress distribution within a rupture figure and in equilibrium with the external forces cannot necessarily be extended to a safe equilibrium stress distribution in the entire soil body. However, an inspection of the proofs of the limit theorems reveals that they are valid within a given rupture figure. Therefore it would be more correct in soil mechanics to say that because the static solution (3) to (5) in the paper satisfies the equation \(W_e = W_d\) for the considered rupture figure, the solution is exact within the considered rupture figure.

Variation of the parameters \(R\) and \(\theta\)

Assume that a carrying capacity analysis is made solely on the basis of the identity \(W_e = W_d\) for the considered failure figure. For given \(V\), \(H\) and \(a\) it is then natural to look for the values of the free parameters \(R\) and \(\theta\) for which there is minimal carrying capacity as expressed by the needed largest cohesion stress \(c\). Extremum is obtained from the conditions

\[
\frac{\partial c}{\partial R} = 0, \quad \frac{\partial c}{\partial \theta} = 0
\]

(4)
which are tantamount to the conditions

\[
\frac{\partial W_e}{\partial R} = \frac{\partial W_d}{\partial R}, \quad \frac{\partial W_e}{\partial \theta} = \frac{\partial W_d}{\partial \theta}
\]

as it directly follows from the fact that \( W_e \) is independent of \( c \) and \( W_d \) is directly proportional to \( c \), (see Eq. (6) in the paper).

These conditions together with \( W_e = W_d \) lead to the equations (7) to (9) in the paper. These equations are different from the static equations (3) to (5) in the paper. Moreover, for given \( H/V, a \) and \( V \) they lead to a value of \( c \) that is larger than the value of \( c \) obtained from the static equations. Thus the upper bound theorem is violated, so where is the error? The error comes from the neglect of the plastic dissipation in the lift off line from B to B’. This dissipation is zero for B and B’ coincident but the partial derivatives are different from zero. The correct plastic dissipation reads (see Appendix 2 of the paper)

\[
cR^2 \left[ \theta + \left( \theta + \frac{1}{2} + \frac{\pi}{4} \right) \tan^2 \theta \right] + c \int_{b}^{b_{\text{max}}} \left[ \sqrt{(\xi - b)^2 + R^2} - (\xi - b) \right] \, d\xi
\]

where \( b_{\text{max}} \) is the width of the physical footing and \( b = R \tan \theta \). Even though the footing moves upward to the left of the point B, there are both shear and compression normal stresses along the lift up line as a consequence of the associated flow rule (see Figure 1 top). Thus the soil material glues to the foundation without separation taking place.

Even though such behavior may be judged to make the chosen plastic model physically unrealistic, it might be reasonable to adopt the plastic model behavior in a neighborhood of the crack tip. For \( b_{\text{max}} - b \leq 0 \) we may then replace the integral term by \( c(b_{\text{max}} - b)R = c(b_{\text{max}} - R \tan \theta)R \) obtained by first order Taylor expansion. With this correction the conditions (5) together with (6) in the paper give equations that are identical to the static equations.

**Some conclusions**

For fixed \( a \) and ratio \( H/V > 0 \) the static solution (3) to (5) of the paper uniquely determines the rupture figure and the ratio \( V/c \).

Under the same conditions there is an infinity of kinematic admissible solutions determined by the work equation (6) in the paper. Both the angle \( \theta \) and the radius \( R \) may be chosen freely as positive functions \( H/V > 0 \). Then substitution into the work equation (6) gives

\[
\frac{V}{c} = \frac{\frac{R}{V} \left( \theta \left( \frac{H}{V} \right) + \left( \frac{\pi}{4} + \frac{1}{2} + \theta \left( \frac{H}{V} \right) \right) \tan^2 \theta \left( \frac{H}{V} \right) \right)}{R \left( \frac{H}{V} \right) \tan \theta \left( \frac{H}{V} \right) - a + \frac{H}{V} R \left( \frac{H}{V} \right)}
\]

(7)

For the particular functions \( R \left( \frac{H}{V} \right) \) and \( \theta \left( \frac{H}{V} \right) \) determined by the static equations (3) to (5) of the paper, this equation gives the same value of \( V/c \) as obtained from (3) of the paper. If the upper bound theorem is valid, \( V/c \) will be larger or equal to the static value for any other choice of the functions \( R \left( \frac{H}{V} \right) \) and \( \theta \left( \frac{H}{V} \right) \). However, the choice corresponding to the equations (7) to (9) gives a value less than the static value. Thus the
The upper bound theorem is not valid. This statement is true because it is not required for the validity of the limit theorems that the rupture figure corresponding to the static solution is the same as the rupture figure corresponding to the kinematic solution. The set of all rupture figures of the specified geometric form defines a consistent system within which the limit theorems are valid if separation does not take place in the rupture lines. This is analogous to the property of the plastic yield hinge model in frame structures. As long as the hinges possess full rotation capacity the limit theorems are valid.

The reason for the demonstrated invalidity of the upper bound theorem in the present example is that the boundary of the plastic body is not fixed. To make the modeling consistent with the yield condition (the part of it defined by $(\sigma - c)^2 + \tau^2 = c^2$ for $\sigma \leq c$) and the associated flow rule, a rupture line from the left edge of the footing to the point B should be included in the rupture figure. Then the upper bound theorem is necessarily valid and the work equation gets the form as determined by the plastic dissipation (6) above. However, by including this rupture line the static solution should be modified by including the stresses (see Figure 1a) that acts upwards on the footing in this rupture line. It is not easy to determine these stresses from Kötter’s equation that becomes complicated as a consequence of the nonlinear relation between $\sigma$ and $\tau$ in the rupture line. Possibly there is a stress discontinuity at B when passing along the rupture line from the left of B to the right of B. Otherwise Kötter’s equation should be changed to correspond to the circular part of the yield condition in an unknown part of the circular rupture line in a neighborhood of B.

The problem is avoided by deeming the plastic theory as unrealistic when applied over the entire length of the rupture line to the left of B. A reasonable model is obtained if $b_{\text{max}}$ is chosen as the unique value of $b$ obtained from (3) to (5) in the paper for $H/V$ given. For the upper bound theorem to be valid the plastic dissipation part of the work equation must contain the integral term written out in the above expression (6).

In case of inhomogeneous soil the static solution cannot be easily obtained by Kötter’s equation. However, the kinematic upper bound method is still applicable through the iterative procedure indicated in Section 2 of the paper.

**Misprint in the paper**

Eq. (13) should read

$$\frac{\partial}{\partial x} W_d(BB') = c_u [R - 2\sqrt{(b-x)^2 + R^2} + 2(b-x)] \to -c_u R$$

as $x \to b$. 