SCIENTIFIC NOTES

Starting with this issue shorter scientific notes will be published in a special section of BIT under the heading above. Authors are invited to submit contributions to this section concerning theorems, methods, or special tricks which

a) have proved to be more convenient than standard methods in some special applications to be described,

b) have not been observed sufficiently in the literature of elementary numerical analysis or programming,

c) are too elementary or not sufficiently original to motivate a special article,

d) can be described in not more than 1–2 pages.

Such scientific notes should be written in English. Reprints will be furnished as for ordinary papers.

A REMARK ON THE LAGRANGIAN REMAINDER IN TAYLOR'S FORMULA

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In Taylor's formula

$$f(x) = \sum_{i=0}^{n-1} \frac{1}{i!} f^{(i)}(0)x^i + \frac{1}{n!} f^{(n)}(\theta x)x^n, \quad \theta \in ]0,1[$$

with the Lagrangian remainder, it may be worth while to notice the classical result

$$\lim_{x \to 0} \frac{\theta}{n+1}$$

when \(f \in C^{n+1}([a,b]), a < 0, b > 0\) and \(f^{(n+1)}(0) \neq 0\).

The proof follows directly using first the mean value theorem on \(f^{(n)}(\theta x) - f^{(n)}(0)\) and then Taylor's formula with \(n + 2\) terms, and comparing the results. For the special case \(n=1\) the proof is given in [1] § 30, ex. 19.

In this way we obtain

$$\theta = \frac{1}{n+1} f^{(n+1)}(\theta_2 x), \quad \theta_1, \theta_2 \in ]0,1[$$

and hence \(\theta = \frac{1}{n+1} + \varepsilon_1(x)\), where \(\varepsilon_1(x) \to 0\) for \(x \to 0\).
A further application of the mean value theorem gives the limiting formula
\[ \frac{1}{n!} f^{(n)}(\theta x) x^n = \frac{1}{n!} f^{(n)} \left( \frac{x}{n+1} \right) x^n + x^{n+1} \varepsilon(x) \] (2)
where
\[ \varepsilon(x) = \frac{1}{n!} f^{(n+1)} \left( \frac{x}{n+1} + \theta_3 x \varepsilon_1(x) \right) x \varepsilon_1(x), \quad \theta_3 \in ]0,1[. \]

If a Lipschitz condition is valid for \( f^{(n+1)} \), i.e.
\[ |f^{(n+1)}(x_1) - f^{(n+1)}(x_2)| \leq L|x_1 - x_2|, \quad x_1, x_2 \in [a,b], \]
it follows from (1) and \( \varepsilon_1(x) = \theta - \frac{1}{n+1} \) that
\[
|\varepsilon(x)| = \frac{\left| f^{(n+1)} \left( \frac{x}{n+1} + \theta_3 x \varepsilon_1(x) \right) \right|}{f^{(n+1)}(\theta_4 x)} \frac{\left| f^{(n+1)}(\theta_4 x) - f^{(n+1)}(\theta_4 0) \right|}{(n+1)!} \leq |1 + \varepsilon_2(x)| \frac{L|x|}{(n+1)!}
\]
where \( \varepsilon_2(x) \to 0 \) for \( x \to 0 \).

By using the value \( \theta = \frac{1}{n+1} \) in the Lagrangian remainder we find that the error involved is of the order of magnitude of at least \( x^{n+2} \), i.e. \( x^{n+1} \varepsilon(x) = O(x^{n+2}) \). Any other choice in general will give an error of the order of magnitude \( x^{n+1} \).

Example
An algorithm which may be used to obtain numerical solutions of a differential equation of the form \( y'' = f(x,y) \) and which has been proposed by R. de Vogelaere [2] on the basis of Newton’s interpolation formula provides an example of the application of (2).

Putting \( \theta = \frac{1}{3} \) in Taylor’s formula for \( n = 2 \) we get
\[ y(x+h) = y(x) + y'(x)h + \frac{1}{2} y''(x)h^2 + h^3 \varepsilon(h) . \]

If \( y'' \) satisfies a Lipschitz condition, the error introduced at the step from \( x \) to \( x+h \) is \( O(h^4) \). Linear extrapolation and interpolation give
\[
\begin{align*}
y''(x + \frac{1}{3} h) &= \frac{1}{3} \bigl( 4y''(x) - y''(x - h) \bigr) + O(h^3) \\
y''(x + \frac{1}{3} 2h) &= \frac{2}{3} \bigl( y''(x) + 2y''(x + h) \bigr) + O(h^2)
\end{align*}
\]

With \( y_n = y(x_0 + nh), z_n = y'(x_0 + nh), f_n = f(x_0 + nh, y_n) \) we then have the algorithm
\[
\begin{align*}
y_{n+1} &= y_n + z_n h + \frac{1}{6} (-f_{n-1} + 4f_n) h^2 + O(h^4), \\
y_{n+2} &= y_n + 2z_n h + \frac{3}{2} (f_n + 2f_{n+1}) h^2 + O(h^3), \\
z_{n+2} &= z_n + \frac{1}{6} (f_n + 4f_{n+1} + f_{n+2}) h + O(h^3)
\end{align*}
\]

where the last recursion formula is simply Simpson's formula. We observe that R. de Vogelaere's method is a two step method, which promises a good accuracy in spite of its simplicity. However, the stability properties of the method have to be investigated separately.

REFERENCES


DANMARKS INGENIØRAKADEMI
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