Gamma type random field for silo pressure *

Ove Ditlevsen†

Abstract. Measurements show that silo wall pressures exhibit large fluctuations in time and space during discharge of the silo. This observation is important for the design of the silo wall because spatial pressure variations may impose substantial bending moments that otherwise may be small or vanishing due to the carrying ability of the membrane forces in the silo wall. Information about the stochastic properties of this pressure variation cannot be obtained from any existing continuum model for the silo medium flowing within the confinement of the silo walls. Therefore the modeling must presently be tied to statistical analysis of the empirical evidence combined with simple mechanical principles. This paper summarizes the mathematics of a gamma distribution type random field model resulting from the data analysis and discusses the implications for the reliability analysis of the silo wall.

Silo load model. A stochastic silo load model has been formulated in (Ditlevsen & Berntsen 1999) on the basis of pressure cell data obtained during discharge plug flow of barley in a 7 m internal diameter, 40 m high circular concrete silo in Karpalund, Sweden. The statistical analysis of the data combined with equilibrium considerations lead to a stochastic field model that defines the wall pressure $p(z,\theta)$ at the dimensionless level coordinate z (i.e. level/ silo radius a) and the angular coordinate θ. The model applies to levels z both sufficiently deep below the medium surface and sufficiently high above the outlet to allow an assumption of statistical homogeneity along a vertical. In the upper part of the medium the field can by a corrective factor be modified to obey Janssens wall pressure formula in the mean such that the pressure becomes zero at the medium surface. The pressure field model is

$$p(z,\theta)=\int_{\mathbb{R}^2}\left[\frac{1}{\beta^2}\varphi\Big(\frac{\zeta-z}{\beta}\Big)\varphi\Big(\frac{\omega-\theta}{\beta}\Big)+\frac{e^{\alpha^2/2}}{\alpha^2+\beta^2}\varphi\Big(\frac{\zeta-z}{\sqrt{\alpha^2+\beta^2}}\Big)\varphi\Big(\frac{\omega-\theta-\pi}{\sqrt{\alpha^2+\beta^2}}\Big)\right]G(\zeta,\omega)d\zeta d\omega \quad (1)$$

where $\varphi(x)=\exp(-x^2/2)/\sqrt{2\pi}$, $\alpha=0.401$, and $\beta=0.058$. The function $G(z,\theta)$ is a gamma distribution generated white noise field of mean $\mu(z,\theta)$ and intensity $I(z,\theta)$. This white noise field is formally defined in its integrated form by the limit-in-distribution operation

$$\int_{z_1}^{z_2}\int_{\theta_1}^{\theta_2}G(\zeta,\omega)\,d\zeta d\omega=\lim_{\max_{i,j}\{\Delta\zeta_i,\Delta\omega_j\}\to 0}\sum_i\sum_j\Gamma(\zeta_i,\omega_j;\zeta_{i+1},\omega_{j+1}) \quad (2)$$

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†Technical University of Denmark, DK 2800 Lyngby, Denmark, http://www.bkm.dtu.dk

where the increments $\Delta\zeta_i = \zeta_{i+1} - \zeta_i$, $\Delta\omega_j = \omega_{j+1} - \omega_j$ correspond to subdivisions $\zeta_0 = z_1 < \zeta_1 < \ldots < \zeta_{m-1} < \zeta_m = z_2$ and $\omega_0 = \theta_1 < \omega_1 < \ldots < \omega_{n-1} < \omega_n = \theta_2$ of the intervals $[z_1, z_2]$ and $[\theta_1, \theta_2]$, respectively, and where $\Gamma(\zeta_i, \omega_j; \zeta_{i+1}, \omega_{j+1})$ are gamma-distributed random variables with parameters $k = \Delta\zeta_i\Delta\omega_j\mu(\zeta_i, \omega_j)^2/I(\zeta_i, \omega_j)$ and $\lambda = \mu(\zeta_i, \omega_j)/I(\zeta_i, \omega_j)$ in the gamma density $f(x) \propto x^{k-1}\exp(-\lambda x)$, $x \geq 0$.

The equilibrium argument behind the model (1) is as follows. Assume that the shear stresses between the medium and the silo wall act vertically during the plug flow. A unit compression force imposed to act through the thin boundary layer between the wall and the solid medium plug at the point (z, θ) of the silo wall can be assumed to be equilibrated by a less concentrated reactive normal stress field acting on the opposite wall. Even though this reactive pressure distribution is unknown, it may be sufficient for engineering purposes to represent its value at the position (y, v) by the plausible one-parameter family of rotation symmetric functions

$$\frac{e^{\alpha^2/2}}{\alpha^2}\varphi\Big(\frac{y-z}{\alpha}\Big)\varphi\Big(\frac{v-\theta-\pi}{\alpha}\Big) \tag{3}$$

By projection of the pressure defined by (3) on the direction of the unit force followed by integration over the cylinder surface each function in this family can be shown to equilibrate the imposed unit force asymptotically for small values of the parameter α. By the statistical analysis it is experienced that the empirical distributions of the pressures obtained at each individual pressure cell is well fitted by a gamma distribution. This observation suggests a mathematical stochasticity source of the form

$$X(z, \theta) = \frac{1}{\beta^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi\Big(\frac{\zeta-z}{\beta}\Big)\varphi\Big(\frac{\omega-\theta}{\beta}\Big)G(\zeta, \omega)\,d\zeta d\omega \tag{4}$$

where β is a suitably small constant and where $G(z, \theta)$ is an inhomogeneous 2π-periodic in θ, delta-correlated and gamma-distributed random field of mean $E[G(z, \theta)] = \mu(z, \theta)$ and intensity $I(z, \theta)$. Thus formally $\mathrm{Cov}[G(z_1, \theta_1), G(z_2, \theta_2)] = \sqrt{I(z_1, \theta_1)I(z_2, \theta_2)}\,\delta(z_2 - z_1)\sum_{n=-\infty}^{\infty}\delta(\theta_2 - \theta_1 + 2n\pi)$, where $\delta(\cdot)$ is Dirac's delta function. The random pressure field $p(z, \theta)$ is then modeled by superposition as

$$p(z, \theta) = X(z, \theta) + \frac{e^{\alpha^2/2}}{\alpha^2}\int_{-\infty}^{\infty}\varphi\Big(\frac{y-z}{\alpha}\Big)\,dy\int_{-\infty}^{\infty}X(y, v)\varphi\Big(\frac{v-\theta-\pi}{\alpha}\Big)\,dv \tag{5}$$

from which (1) is obtained. The corresponding expectation function is obtained by substituting $\mu(\zeta, \omega)$ for $G(\zeta, \omega)$. Under the assumption that $\mu(\zeta, \omega)$ is slowly varying on the scale of α and β we get the approximation $E[p(z, \theta)] \approx \mu(z, \theta) + e^{\alpha^2/2}\mu(z, \theta + \pi)$. Restricting $E[p(z, \theta)]$ to be periodic with period π ensuring that $E[p(z, \theta)]$ satisfies self-equilibrium along the perimeter, we get

$$\mu(z, \theta) \approx E[p(z, \theta)]/(1+e^{\alpha^2/2}) \tag{6}$$

Assuming that $\beta << \alpha$ and that the relative variation of $I(z, \theta)$ is small over increments of $\sqrt{z^2 + \theta^2}$ of order of size α, and moreover that $\varphi(\pi/\alpha)$ is sufficiently small compared to 1 to be discarded, the following approximate relation between the intensity $I(z,\theta)$ and the variance $\mathrm{Var}[p(z,\theta)]$ is obtained:

$$I(z,\theta) \approx \mathrm{Var}[p(z,\theta)]\, 4\pi\beta^2/[1+(\beta/\alpha)^2 e^{\alpha^2}] \tag{7}$$

For sufficiently large $H = z_2 - z_1$ compared to α and β the average pressure $\bar{p}(\theta) = \frac{1}{H}\int_{z_1}^{z_2} p(z,\theta)\,dz$ at θ over the height H can asymptotically as $\beta \to 0$ be approximated by

$$\bar{p}(\theta) \approx \int_{-\infty}^{\infty} \Big[\frac{1}{\beta}\varphi\Big(\frac{\omega-\theta}{\beta}\Big)+\frac{e^{\alpha^2/2}}{\sqrt{\alpha^2+\beta^2}}\varphi\Big(\frac{\omega-(\theta+\pi)}{\sqrt{\alpha^2+\beta^2}}\Big)\Big]\Big[\frac{1}{H}\int_{z_1}^{z_2} G(\zeta,\omega)\,d\zeta\Big]d\omega \tag{8}$$

with physical dimension of force per area unit. If $\mu(z;\theta)$ and $I(z;\theta)$ do not depend on z, the integral average $\bar{G}(\omega) = \frac{1}{H}\int_{z_1}^{z_2} G(\zeta,\omega)\,d\zeta$ is a gamma-distributed white noise field with respect to ω of mean $\mu(\omega)$ and intensity $I(\omega)/H$.

For the determination of the internal forces in the (elastic) silo wall it is convenient to use the Fourier series representation $\bar{p}(\theta) = \sum_{-\infty}^{\infty} C_n e^{in\theta}$ in which the coefficient sequence $\{C_n\}$ asymptotically as $\beta \to 0$ is given by

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\bar{p}(\theta)e^{-in\theta}d\theta \approx e^{-(n\beta)^2/2}\Big[1+(-1)^n e^{-(n^2-1)\alpha^2/2}\Big]\frac{1}{2\pi}\int_{-\pi}^{\pi}\bar{G}(\omega)e^{-in\omega}d\omega \tag{9}$$

Inhomogeneity. The data show that the pressure field is not homogeneous along the perimeter of the silo. The data from each pressure cell are well fitted by a gamma distribution with density $f(x) \propto x^{k-1}\exp(-\lambda x)$, $x \geq 0$. The maximum likelihood estimates of the parameters k and λ show considerable variability beyond what is generated by statistical uncertainty, however. If the estimates of the mean k/λ and variance k/λ^2 are assumed to be realizations of values of a pair of homogeneous cross-correlated lognormal random fields with given parametric correlation structure defined by a minimal number of parameters, and superposed by quantitatively given independent statistical uncertainty, then it is possible to estimate the means, variances, and correlation parameters of these fields by the method of maximum likelihood. These fields can then be used to define the interpolating conditional fields given the estimates at the pressure cells. This interpolating conditional field may be taken as completely defining the stochastic wall pressure field in the silo in which the measurements have been made. The estimated homogeneous pair of fields may in lack of observations from several similar silos be introduced as a code specified generator of load fields applicable to concrete silos similar to the Karpalund silo and with barley as medium. The consequences of this modeling is a considerable increase of the bending moments compared to what is considered in current design practice. Even for the interpolating load field failure of the Karpalund silo wall becomes predicted with high probability (Ditlevsen 1999). In fact, during operation severe vertical cracks developed in the Karpalund silo making a repair action necessary. The silo wall thickness has been increased from originally 0.18 m to 0.30 m with simultaneous considerable increase of the circumferential reinforcement.

The homogeneous Gaussian vector field $(M(\omega,z),\, V(\omega,z)) = (\log E[p(\omega,z)],\, \log \mathrm{Var}[p(\omega,z)])$ is modeled to the maximal degree of simplicity without loosing its ability to capture the essential variability features of the mean and the variance over the silo wall surface (Ditlevsen 1999). The field is completely defined by $E[(M(\omega), V(\omega))] = (\mu_m, \mu_v)$, and

$$\text{Cov}[M(\omega_1, z_1), M(\omega_2, z_2)] = \sigma_m^2[a_{11}+c_{11}\cos 2(\omega_1-\omega_2)]\, r(z_1-z_2) \tag{10}$$

$$\text{Cov}[M(\omega_1, z_1), V(\omega_2, z_2)] = \sigma_m\sigma_v\rho[a_{12}+c_{12}\cos 2(\omega_1-\omega_2)]\, r(z_1-z_2) \tag{11}$$

$$\begin{aligned}\text{Cov}[V(\omega_1, z_1), V(\omega_2, z_2)] = \sigma_v^2[&a_{22}+b_{22}\cos(\omega_1-\omega_2) \\ &+c_{22}\cos 2(\omega_1-\omega_2)+d_{22}\cos 3(\omega_1-\omega_2)]\, r(z_1-z_2)\end{aligned} \tag{12}$$

$$r(z) = \exp[-(\pi/4)(z/R)^2] \tag{13}$$

where $a_{11}, \ldots, d_{22} \geq 0$, $a_{11}{=}1 - c_{11}$, $a_{22}{=}1 - b_{22} - c_{22} - d_{22}$, $a_{12}{=}1 - c_{12}$; $c_{12}{=}\sqrt{c_{11}c_{22}}/(\sqrt{a_{11}a_{22}} + \sqrt{c_{11}c_{22}})$, $\rho^2 \leq (\sqrt{a_{11}a_{22}} + \sqrt{c_{11}c_{22}})^2$, $a_{11}{=}0.318$ $c_{11}{=}0.682$, $a_{22}{=}0.175$, $b_{22}{=}0.105$, $c_{22}{=}0.556$, $d_{22}{=}0.154$, $a_{12}{=}0.277$, $c_{12}{=}0.723$, $(\mu_m, \mu_v) = (3.52, 4.98)$ [+ log(kPa)], $\sigma_m = 0.207$, $\sigma_v = 0.695$, $\rho = 0.497$, and $R = 0.574$.

Obviously this Gaussian vector process $(M(\omega, z), V(\omega, z))$ can be defined explicitly in terms of a set of 10 independent Gaussian fields $U_1(z), \ldots, U_{10}(z)$ all of mean zero, standard deviation one, and correlation function (13):

$$\begin{aligned}[M(\omega,\cdot)-\mu_m]/\sigma_m &= [\sqrt{1+k\rho}\,U_1+\sqrt{1-k\rho}\,U_2]\sqrt{a_{11}/2} \\ &+([\sqrt{1+k\rho}\,U_3+\sqrt{1-k\rho}\,U_4]\cos 2\omega + [\sqrt{1+k\rho}\,U_5+\sqrt{1-k\rho}\,U_6]\sin 2\omega])\sqrt{c_{11}/2}\end{aligned} \tag{14}$$

$$\begin{aligned}[V(\omega,\cdot)-\mu_v]\sigma_v &= [\sqrt{1+k\rho}\,U_1-\sqrt{1-k\rho}\,U_2]\sqrt{a_{22}/2} + [U_7\cos\omega+U_8\sin\omega]\sqrt{b_{22}} \\ &+([\sqrt{1+k\rho}\,U_3-\sqrt{1-k\rho}\,U_4]\cos 2\omega + [\sqrt{1+k\rho}\,U_5-\sqrt{1-k\rho}\,U_6]\sin 2\omega)\sqrt{c_{22}/2} \\ &+[U_9\cos 3\omega+U_{10}\sin 3\omega]\sqrt{d_{22}}\end{aligned} \tag{15}$$

$$k = (\sqrt{a_{11}a_{22}} + \sqrt{c_{11}c_{22}})^{-1} \tag{16}$$

Reliability analysis. Due to the representations (14) and (15) it is easy to simulate realizations of $(M(\omega, z), V(\omega, z))$. Each realization defines a random wall pressure field for which the reliability (i.e. the survival probability) of the silo shell may be evaluated by a finite element calculation, e.g. by use of the Fourier representation (8). This calculated conditional reliability is next raised to a power that corresponds to the number of independent replacements of the silo load in the same silo during a given reference period. By averaging over a suitably large number realizations of $(M(\omega, z), V(\omega, z))$ the reliability of a considered silo design is finally evaluated. This topic is treated in (Ditlevsen 1999), in which it is shown that it is sufficiently accurate for the reliability analysis to assume that the conditional internal force fields in the silo wall are Gaussian fields.

References

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