Positive random fields for modeling material stiffness and compliance

A.M.Hasofer
Victoria University of Technology, Australia

O. Ditlevsen, and N.J. Tarp-Johansen
Department of Structural Engineering and Materials, Technical University of Denmark, Build. 118, DK 2800 Lyngby, Denmark

ABSTRACT: Positive random fields with known marginal properties and known correlation function are not numerous in the literature. The most prominent example is the lognormal field for which the complete distribution is known and for which the reciprocal field is also lognormal. It is of interest to supplement the catalogue of positive fields beyond the class of those obtained by simple marginal transformation of a Gaussian field, this class containing the lognormal field. As a minimum for a random field to be included in the catalogue it is required that an algorithm for simulation of realizations can be constructed and that the one-dimensional marginal moments up to at least the fourth order as well as the correlation function of the field are given explicitly. This information makes it possible to use and check the so-called Winterstein approximations for finite element calculations of structures with material properties modeled in terms of the considered random fields. The paper adds the gamma field, the Fisher field, the beta field, and their reciprocal fields to the catalogue. These fields are all defined on the basis of sums of squares of independent standard Gaussian random variables. All the existing marginal moments and the correlation functions are obtained explicitly. Also an inverse Gaussian field is added to the catalogue. It is defined in terms of first passage times in correlated joint Brownian motions. Finally an n-dimensional random vector of positive components is defined such that it can be used as an approximation to a discretization of a homogeneous Gaussian field with any specified correlation function and coefficient of variation less than $1/\sqrt{n}$.

1 INTRODUCTION

Random field modeling of material parameters such as, for example, the material stiffness (Young's modulus) or its reciprocal, the compliance, presents a problem of choosing the input parameters among several equivalent parameters. Structural finite element mechanics provide good arguments for choosing the bending compliance of a beam as input rather than the bending stiffness, [5]. However, it seems reasonable to avoid that the stiffness field corresponding consistently to a chosen mathematically well-behaved compliance field should be an ill-behaved field for which even the mean may not exist. For example, if the moments up to at least order n are required to exist for both fields, then any of the two density functions must have lower tails that as functions of x approach zero at least as fast as $x^{n-1+\epsilon}$ as $x \to 0$ and upper tails that approach zero at least as fast as $x^{-(n+1+\epsilon)}$ as $x \to \infty$ for some $\epsilon > 0$.

Even though it is not a necessary condition, it is particularly convenient to have a field model for which the reciprocal field belongs to the same distribution type as the field itself and even has the same relative moments (i.e. coefficient of variation, skewness, kurtosis,..., correlation (coefficient) function, etc.). Moreover, if explicit formulas for these relative moments can be given, it enhances the possibilities of calculating sufficiently accurate approximate values of the relative moments of any set of weighted integrals of the field or of the reciprocal of the field. Such weighted integrals play an important role in the formulation of stochastic finite element methods [2, 5].

Noting that any power of a lognormal field is a lognormal field it is easily seen that the lognormal field model possesses all these wanted properties. This paper lists some other fields that possess all or some of the wanted properties and that may be useful as models for material parameter fields.

The most obvious idea is to take the field to be the ratio of two mutually independent and identically distributed nonnegative fields. However, it is not so obvious how to choose these fields such that an explicit one-dimensional marginal distribution of the ratio between them is obtained and such that all the wanted relative moments and the correlation coefficient function can be given explicitly. One successful choice is a gamma field defined as the sum of an even number of squares of mutually independent and identically distributed zero mean Gaussian fields. Then the marginal distribution of the ratio becomes Fisher's z-distribution. All finite marginal moments and the correlation coefficient function can be obtained explicitly. The same holds for the beta field defined as the ratio between a Fisher field and 1 plus the same Fisher field, and obviously also for the reciprocal of a beta field, because this is simply the constant 1 added to a Fisher field.

In proc. of ICOSSAR'97, Kyoto, Nov. 24–28, 1997, pp. 723–730, A.A. Balkema, Rotterdam, 1998

Another candidate for a nonnegative field is the inverse Gaussian field. It is defined by letting all the finite-dimensional sets of random variables be the sets of first zero crossing times of a corresponding set of equally distributed mutually dependent Brownian motions that all have drift -1 and diffusion coefficient σ. The joint Brownian motions start at zero time from the common value 1. Their mutual dependency is induced solely by having correlation between simultaneous increments. The moments of any order exist for the inverse Gaussian field defined in this way and for its reciprocal field. The marginal moments are well known. The correlation function of the field is obtained by solving a corresponding boundary value problem for the relevant diffusion equation derived from the backward Kolmogorov equation for a correlated pair of Brownian motions. For the reciprocal field to an inverse Gaussian field the correlation function is not known. However, numerical results can be obtained by simulation or by use of Winterstein approximation technique and numerical integration [10, 4, 3, 7].

Aiming at constructing an example of a nonnegative homogeneous field with a somewhat peculiar structure, it is suggested in [6] to model the bending stiffness of a beam in the following way. Assume that each bivariate density can be defined in normalized form to be constant inside an ellipse and zero outside the ellipse such that the one-dimensional densities become proportional to $(1-x^2)^{1/2}$, $x^2 \leq 1$. The ellipse is then determined by the correlation coefficient ρ. This ellipse is coincident with a constant probability density ellipse of the standardized bivariate normal density with the same correlation coefficient ρ. It is shown in the present paper that consistent joint distributions of higher order than four do not exist. However, it is possible to make a generalization of the iso-density curve coincidence to the distributions of order up to any chosen finite order m. For any homogeneous Gaussian field and for any set of m equidistant points it is shown that there is an m-dimensional random vector for which the $(m-2)$-dimensional distributions are uniform on the hyperellipsoids inscribed in the box $[-1,1]^{m-2}$, and such that the vector has the same correlation coefficient matrix as the Gaussian field restricted to the chosen set of m points. The one-dimensional distribution of the vector is proportional to $(1-x^2)^{(m-3)/2}$, $x^2 \leq 1$. By a suitable linear marginal transformation a vector with identically distributed positive components is obtained. If m is sufficiently large the vector can be interpreted as an approximate discretization of the corresponding Gaussian field, and in the limit $m \to \infty$ it becomes equivalent to the Gaussian field. The corresponding reciprocal fields are less simple.

The different types of fields listed in the paper are illustrated by examples of realizations.

2 GAMMA FIELDS, FISHER FIELDS, AND BETA FIELDS

Various nonnegative homogeneous fields (or processes, if t is time) can be constructed from a collection of $2(m+n)$ independent zero mean unit variance Gaussian fields U_r, $r = 1,\ldots,2(m+n)$, with the same correlation function $\rho(t_2-t_1) = \text{corr}[U_r(t_1), U_r(t_2)]$. The two fields

$$X(t) = \frac{1}{2}\sum_{r=1}^{2m} U_r^2(t), \quad Y(t) = \frac{1}{2}\sum_{r=2m+1}^{2(m+n)} U_r^2(t) \quad (1)$$

may be called a homogeneous Gamma(m) field and Gamma(n) field, respectively, because their one-dimensional marginal distributions are gamma distributions of mean m and n, respectively, i.e. $X(t)$ has the density function

$$f_X(x) = \frac{1}{\Gamma(m)} x^{m-1} e^{-x}, \quad x > 0 \quad (2)$$

and similarly for $Y(t)$. The moment of order β is

$$E[X^\beta] = \frac{\Gamma(m+\beta)}{\Gamma(m)} = \prod_{i=0}^{\beta-1}(m+i), \quad \beta > -m \quad (3)$$

and, in particular, $E[X] = \text{Var}[X] = m$. The correlation function is

$$\text{corr}[X(t_1), X(t_2)] = [\rho(t_2-t_1)]^2 \quad (4)$$

which may be shown by direct calculation, or by using the moment generating function (15) below. Setting β to a negative integer in (3) it follows that the reciprocal field $1/Y(t)$ to the homogeneous Gamma(n) field $Y(t)$ has finite moments up to order $n-1$ only. The correlation function $\text{corr}[1/Y(t_1), 1/Y(t_2)]$ is given by (19).

The homogeneous Fisher(m,n) field $Z(t)$ is defined as the ratio

$$Z(t) = \frac{X(t)}{Y(t)} \quad (5)$$

for which the one-dimensional marginal density function is Fisher's density

$$f_Z(x) = B(m,n)^{-1}\frac{x^{m-1}}{(1+x)^{m+n}}, \quad x > 0 \quad (6)$$

where $B(m,n) = \Gamma(m)\Gamma(n)/\Gamma(m+n)$ is the beta function. All moments up to order $n-1$ are finite. The moment of order β is

$$E[Z^\beta] = \frac{\Gamma(m+\beta)\Gamma(n-\beta)}{\Gamma(m)\Gamma(n)} \quad (7)$$

In particular, $E[Z] = m/(n-1)$ and $\text{Var}[Z] = m(m+n-1)/(n-1)^2(n-2)$. Obviously the reciprocal field $1/Z(t)$ is the homogeneous Fisher(n,m) field.

The homogeneous Beta(m,n) field $B(t)$ is defined as

$$B(t) = \frac{Z(t)}{1+Z(t)} = \frac{X(t)}{X(t)+Y(t)} \tag{8}$$

with the one-dimensional marginal density function being the beta density

$$f_B(x) = B(m,n)^{-1}x^{m-1}(1-x)^{n-1} \tag{9}$$

for which the moment of order β is

$$E[B^\beta] = \frac{\Gamma(m+\beta)\Gamma(m+n)}{\Gamma(m)\Gamma(m+\beta+n)} \tag{10}$$

In particular, $E[B] = m/(m+n)$ and $\text{Var}[B] = mn/(m+n)^2(m+n+1)$. The reciprocal field is $1/B(t) = 1 + 1/Z(t)$, that is, the homogeneous Fisher(n,m) field $1/Z(t)$ added to the constant 1. To derive the correlation functions for the two fields $Z(t)$ and $B(t)$ (and for the corresponding reciprocal fields) let $X_1 = X(t_1)$, $Y_1 = Y(t_1)$, $X_2 = X(t_2)$, $Y_2 = Y(t_2)$. Defining the moment generating functions

$$\begin{aligned} M_Z(\theta_1,\theta_2,\theta_3,\theta_4) &= \\ E[e^{\theta_1 X_1+\theta_2 X_2-\theta_3 Y_1-\theta_4 Y_2}] \\ &= E[e^{\theta_1 X_1+\theta_2 X_2}]E[e^{-\theta_3 Y_1-\theta_4 Y_2}] \end{aligned} \tag{11}$$

$$\begin{aligned} M_B(\theta_1,\theta_2,\theta_3,\theta_4) &= \\ E[e^{\theta_1 X_1+\theta_2 X_2-\theta_3(X_1+Y_1)-\theta_4(X_2+Y_2)}] \\ &= E[e^{(\theta_1-\theta_3)X_1+(\theta_2-\theta_4)X_2}]E[e^{-\theta_3 Y_1-\theta_4 Y_2}] \end{aligned} \tag{12}$$

it is seen that

$$\int_0^\infty\int_0^\infty M_Z(0,0,\theta_3,\theta_4)d\theta_3 d\theta_4 = E\Big[\frac{1}{Y_1}\frac{1}{Y_2}\Big] \tag{13}$$

$$\int_0^\infty\int_0^\infty \Big[\frac{\partial^2 M}{\partial\theta_1\partial\theta_2}\Big]_{\theta_1=\theta_2=0} d\theta_3 d\theta_4 = \begin{cases} E\Big[\dfrac{X_1}{Y_1}\dfrac{X_2}{Y_2}\Big] \text{ for } M = M_Z \\ E\Big[\dfrac{X_1}{X_1+Y_1}\dfrac{X_2}{X_2+Y_2}\Big] \text{ for } M = M_B \end{cases} \tag{14}$$

It is well-known that (see e.g. [9], pp. 219 and 267)

$$E[e^{\theta_1 X_1+\theta_2 X_2}] = [(1-\theta_1)(1-\theta_2)-\rho^2\theta_1\theta_2]^{-m} \tag{15}$$

in which $\rho = \rho(t_2-t_1)$. Thus

$$\begin{aligned} M_Z(\theta_1,\theta_2,\theta_3,\theta_4) &= \\ [(1-\theta_1)(1-\theta_2)-\rho^2\theta_1\theta_2]^{-m} \\ \cdot[(1+\theta_3)(1+\theta_4)-\rho^2\theta_3\theta_4]^{-n} \end{aligned} \tag{16}$$

$$\begin{aligned} M_B(\theta_1,\theta_2,\theta_3,\theta_4) &= \\ [(1+\theta_3)(1+\theta_4)-\rho^2\theta_3\theta_4]^{-n}[(1-\theta_1+\theta_3) \\ \cdot(1-\theta_2+\theta_4)-\rho^2(\theta_1-\theta_3)(\theta_2-\theta_4)]^{-m} \end{aligned} \tag{17}$$

The integrals in (13) and (2) can be solved analytically by decomposition of the integrands into sums of partial fractions. The results can be expressed in terms of the function

$$\begin{aligned} S_q(x) &= q\Big(\frac{1-x}{-x}\Big)^q\Big[\log(1-x) - \sum_{r=1}^{q-1}\frac{1}{r}\Big(\frac{-x}{1-x}\Big)^r\Big] \\ &= q\sum_{r=0}^\infty \frac{1}{r+q}\Big(\frac{-x}{1-x}\Big)^r, 0 \le x \le 1, q \in \{1,2,\dots\} \end{aligned} \tag{18}$$

with the end values $S_q(0) = 1$ and $S_q(1) = 0$. The correlation functions obtained in this way are

$$\begin{aligned} \text{corr}[Y(t_1)^{-1}, Y(t_2)^{-1}] &= \Big(\frac{n-1}{\rho^2} - n + 2\Big) \\ \cdot\Big[1 - \frac{n-1}{(n-1)-(n-2)\rho^2}S_{n-2}(\rho^2)\Big] \end{aligned} \tag{19}$$

$$\begin{aligned} \text{corr}[Z(t_1), Z(t_2)] &= \\ \frac{(m+\rho^2)(n-1) - m(n-2)\rho^2}{(m+n-1)\rho^2} \\ \cdot\Big[1 - \frac{(m+\rho^2)(n-1)}{(m+\rho^2)(n-1)-m(n-2)\rho^2}S_{n-2}(\rho^2)\Big] \\ \text{both for } n \ge 3 \text{ , and} \end{aligned} \tag{20}$$

$$\begin{aligned} \text{corr}[B(t_1), B(t_2)] = 1 - S_{m+n}(\rho^2) \\ \text{for } n+m \ge 1 \end{aligned} \tag{21}$$

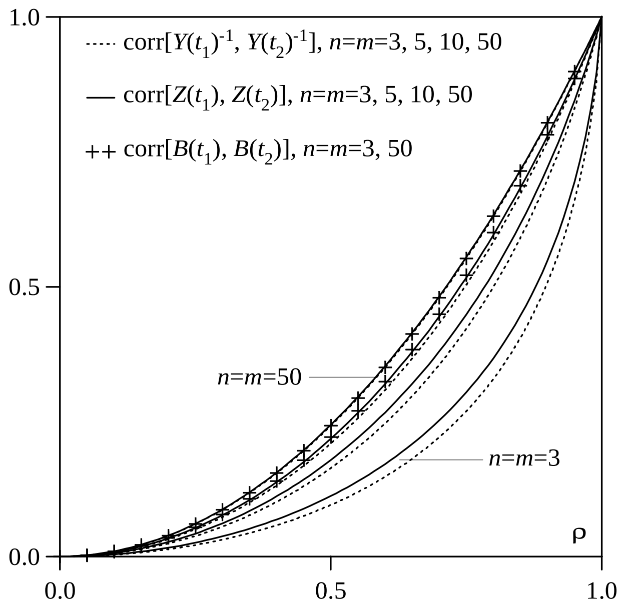

Figure 1: Correlations for the reciprocal Gamma(n), the Fisher(n,m) and the Beta(n,m) fields for $n = m$ as function of the correlation coefficient ρ of the generating Gaussian fields. As $n \to \infty$ the curves approach the correlation ρ^2 of the Gamma(n) field.

It is noteworthy that these correlation coefficients are functions of ρ^2 solely. Thus underlying Gaussian fields with correlation functions that become negative and others for which ρ remains positive will generate the same gamma, Fisher, and beta field as long as they have the same ρ^2. Graphs of examples of the correlation coefficients (19),(20), and (21) as functions of ρ are shown in Fig. 1. Fig. 2 shows examples of realizations of the fields defined in this section.

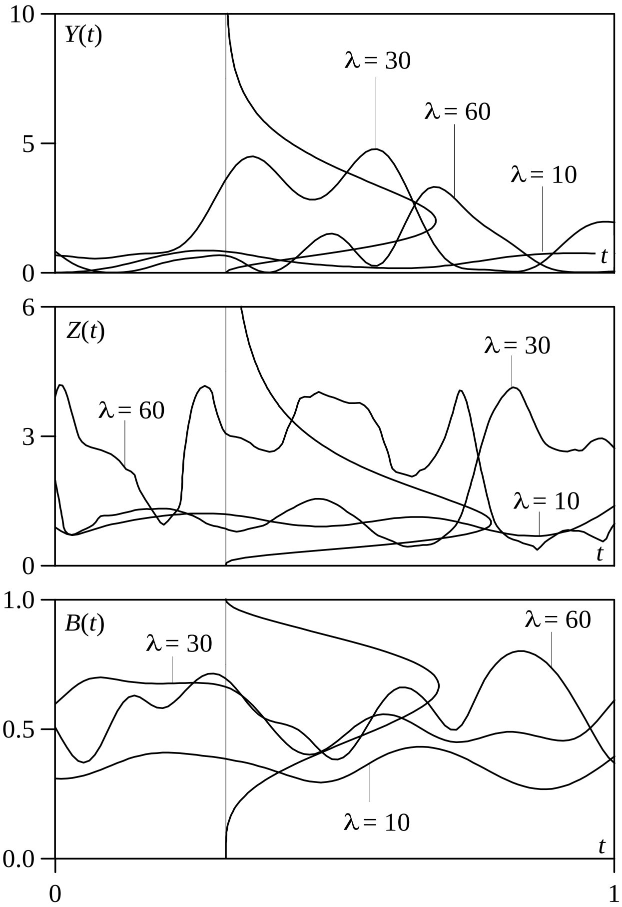

Figure 2: Sample functions of the Gamma(3), the Fisher(5,3) and the Beta(5,3) fields for different correlation lengths. The correlation function for the underlying Gaussian fields is $\rho(t) = \exp(-\lambda t^2)$. The shapes of the density functions are indicated.

3 INVERSE GAUSSIAN FIELDS

Let $W(t;\xi)$ for each fixed value of a spatial coordinate ξ be a Brownian motion (a Wiener process) with the property that the increment $W(t+dt;\xi) - W(t;\xi)$ is a homogeneous Gaussian field in the coordinate ξ with mean $-dt$, variance $\sigma^2 dt$, and correlation function $\rho(\xi_2 - \xi_1)$. Moreover, assume that $W(0;\xi) = 1$ for all ξ. Then the time $T(\xi)$ to first passage of $W(t;\xi)$ through zero is a homogeneous inverse Gaussian field along the ξ-axis. The one-dimensional density is the inverse Gaussian density, see e.g. [8] p.138,

$$f_T(x) = \frac{1}{\sigma x\sqrt{x}}\varphi\Big(\frac{x-1}{\sigma\sqrt{x}}\Big), \quad x > 0 \qquad (22)$$

The density function of the reciprocal $S = 1/T$ is directly obtained as

$$f_S(x) = \frac{1}{\sigma\sqrt{x}}\varphi\Big(\frac{x-1}{\sigma\sqrt{x}}\Big), \quad x > 0 \qquad (23)$$

By comparison of (22) and (23) it is seen that for any $\beta \in \mathbb{R}$

$$E[T^\beta] = \int_0^\infty \frac{x^\beta}{\sigma x\sqrt{x}}\varphi\Big(\frac{x-1}{\sigma\sqrt{x}}\Big)dx = E[S^{\beta-1}] \quad (24)$$

implying that $E[T] = 1$. By differentiation of $E[T^\beta]$ with respect to σ we obtain

$$E[T^{\beta+1}] = \sigma^3\frac{dE[T^\beta]}{d\sigma} + (2+\sigma^2)E[T^\beta] - E[T^{\beta-1}] \qquad (25)$$

By recursive use of this formula we get

$$\begin{aligned} E[T^1] &= 1 \\ E[T^2] &= 1 + \sigma^2 \\ E[T^3] &= 1 + 3\sigma^2 + 3\sigma^4 \\ E[T^4] &= 1 + 6\sigma^2 + 15\sigma^4 + 15\sigma^6 \\ E[T^5] &= 1 + 10\sigma^2 + 45\sigma^4 + 105\sigma^6 + 105\sigma^8 \end{aligned} \qquad (26)$$

The determination of the correlation function of the inverse Gaussian field in terms of the correlation function $\rho(\xi)$ of the Gaussian increment field is a more complicated task. The derivation is given in the appendix. The result is

$$\mathrm{corr}[T(0),T(\xi)] = \rho\Big\{1 + \sigma^2\sqrt{1-\rho^2}e^{2a^2} \cdot \sum_{n=1}^{\infty}(-1)^n n\tan n\phi_0\Big[I_n(2a^2) - \sum_{m=1}^{\infty}\Big(\frac{\sin(n+\nu_m)\phi_0}{(n+\nu_m)\phi_0} + \frac{\sin(n-\nu_m)\phi_0}{(n-\nu_m)\phi_0}\Big)I_{\nu_m}(2a^2)\Big]\Big\} \qquad (27)$$

where

$$a = \frac{1}{\sigma\sqrt{1+\rho}}, \; \phi_0 = \arctan\sqrt{\frac{1+\rho}{1-\rho}} \qquad (28)$$

$\rho = \rho(\xi) > -1$, and I_ν is the νth order modified Bessel function of the first kind. Graphs of $\mathrm{corr}[T(0),T(\xi)]$ as function of $\rho \geq 0$ for different values of $\sigma \geq 1$ are shown in Fig. 3.

The formula (27) is not well suited for numerical evaluation, in particular when $2a^2$ is large. It requires a non-standard high accuracy computer to evaluate $\mathrm{corr}[T(0),T(\xi)]$ for negative values of

ρ and for small values of σ or it requires non-standard computer programming. Fig. 4 shows some examples of realizations of the field $T(\xi)$.

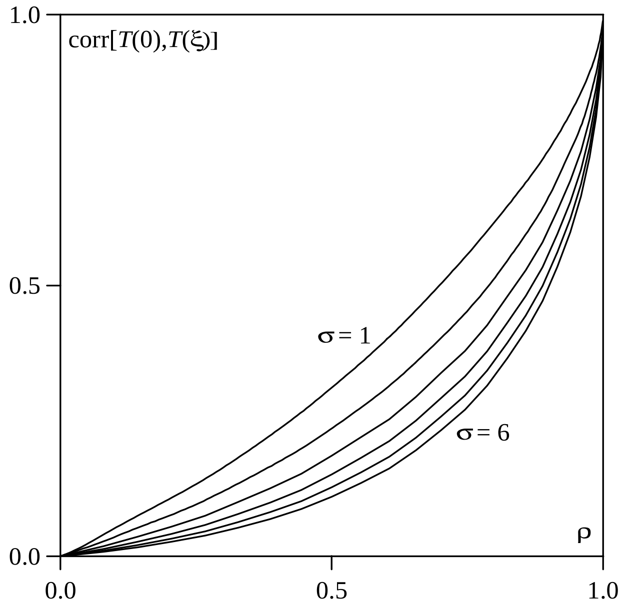

Figure 3: Correlations for the inverse Gaussian field as function of ρ, $\rho \geq 0$, for different variances, $\sigma^2 dt$, of the increments of the underlying Brownian motion.

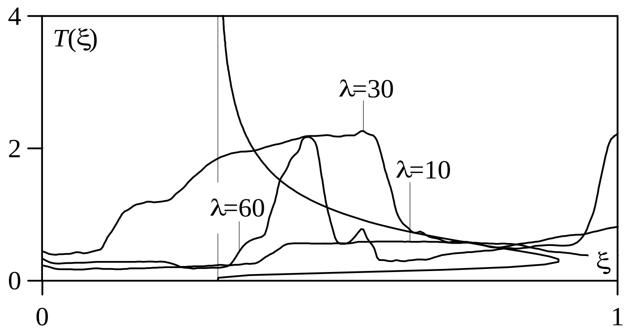

Figure 4: Sample functions of the inverse Gaussian field $T(\xi)$ for different correlation lengths. The correlation function for the increments of the generating Brownian motion is $\rho(\xi) = \exp(-\lambda\xi^2)$. The shape of the density function is indicated.

4 NONNEGATIVE RANDOM VECTOR AS APPROXIMATION TO HOMOGENEOUS GAUSSIAN FIELD

Let the interval $[0, nh]$ on the t-axis be divided into n subintervals of equal length h. To subinterval number i assign a random variable Y_i defined as the ith element of the random vector $\mathbf{Y} = (Y_1, \dots, Y_n)$ where $\mathbf{Y} = \mathbf{AX}$ and $\mathbf{X} = (X_1, \dots, X_n)$ is uniformly distributed on the surface of the zero centered n-dimensional unit sphere. For symmetry reasons the random variables $X_1, \dots, X_n$ are pairwise uncorrelated (but not independent, of course). By suitable choice of the matrix $\mathbf{A}$ the ellipsoidal image of the n-dimensional unit sphere becomes inscribed in the unit box $[-1, 1]^n$. Moreover, $\mathbf{A}$ can be chosen such that $\mathbf{Y}$ gets any specified covariance matrix. In particular the elements of this covariance matrix can be chosen as the covariances of a given homogeneous field restricted to the midpoints of the subintervals. Thus the field defined by $\mathbf{Y}$ can be taken as a second moment approximation to the given field even though this approximation is not a homogeneous field.

Let us first consider the case where $\mathbf{Y} = \mathbf{X}$. Then the joint density of any subset of m of the n random variables in $\mathbf{X}$ has the form

$$f_{\mathbf{X}_m}(\mathbf{x}_m) \propto (\sqrt{1-r^2})^{n-m-2}, \quad r = \| \mathbf{X}_m \| \qquad (29)$$

where the norm is the Euclidian norm. This result is directly proved by induction since the projection of the probability mass given by (29) on any hyperplane through the origin of the m-dimensional space is proportional to

$$\int_u^1 (\sqrt{1-r^2})^{n-m-2} d\sqrt{r^2-u^2} \propto (\sqrt{1-u^2})^{n-m-1} \qquad (30)$$

where u is the distance from the origin on the hyperplane. In particular, if $m = n-1$, the density (29) is consistent with the projection

$$\int_u^1 \delta(r-1) d\sqrt{r^2-u^2} \propto (\sqrt{1-u^2})^{-1} \qquad (31)$$

where $\delta(\cdot)$ is the Dirac delta function. It is seen that if $m = n-2$, then the distribution becomes uniform on the interior of the $(n-2)$-dimensional unit sphere.

Assume now that there is some rotationally symmetric distribution of probability mass inside the n-dimensional unit sphere such that the projection on any $(n-2)$-dimensional subspace becomes a uniform distribution on the interior of the $(n-2)$-dimensional unit sphere (obviously there can only be zero probability mass outside the n-dimensional unit sphere). Then the uniform distribution on the surface of the n-dimensional unit sphere is obtained by moving all the probability mass of the interior to the surface in the positive radial direction. However, then the projected mass on the $(n-2)$-dimensional subspace also moves in the positive radial direction. This implies that the uniform distribution on the interior of the $(n-2)$-dimensional unit sphere cannot be preserved. Thus there is no other distribution than the uniform distribution on the surface of the n-dimensional unit sphere that leads to the derived marginal distributions. Also it is obvious that it is not possible to have a rotationally symmetric distribution in the $(n+1)$-dimensional space that by projection gives a uniform distribution on the surface of the n-dimensional unit sphere. Clearly a projection gives positive density at the origin. The conclusion to be drawn from these considerations is that there is no homogeneous random vector of higher dimension than $m+2$ when it is given that the subvectors of dimension m have uniform distribution on the interior of the m-dimensional unit sphere.

The uniform distribution on the interior of the $(n-2)$-dimensional unit sphere is for any $\mathbf{A}$ carried over to the ellipsoid image of the sphere. Since the image of the n-dimensional unit sphere is inscribed in the n-dimensional unit box, the one-dimensional marginal distributions are not affected by the mapping. According to (29) these are

$$f_{Y_i}(y) \propto (\sqrt{1-y^2}\,)^{n-3}, \quad y \in]-1,1[\,,\ n \geq 2 \tag{32}$$

Now let $Z_i = \sqrt{n}Y_i$. Then

$$f_{Z_i}(z) \propto (\sqrt{1-z^2/n}\,)^{n-3}$$
$$= \exp\Big[\frac{n-3}{2}\log(1-\frac{z^2}{n})\Big] \to e^{-\frac{1}{2}z^2} \tag{33}$$

as $n \to \infty$. A similar limit result is obtained for any of the finite dimensional marginal distributions of fixed dimension. Thus the field defined by $\mathbf{Y}$ converges in distribution to the Gaussian field corresponding to the limit covariance function as $n \to \infty$.

For any sufficiently large n the field defined by $\mathbf{Y}$ is thus an approximation to the corresponding homogeneous Gaussian field, the covariance matrix of $\mathbf{Y}$ being the restriction of the covariance function of the field to the points $h, 2h, \ldots, nh$. The advantage of the approximation is that it can be bounded away from zero by making a suitable translation.

For any matrix $\mathbf{B}$ of type (m,n), $m \leq n$, and rank m the linear mapping $\mathbf{BY}$ defines a random vector situated in an m-dimensional subspace of $\mathbb{R}^n$. This vector is obtained directly from the random unit vector $\mathbf{X}$ by the composite mapping $\mathbf{BY}$ = $(\mathbf{BA})\mathbf{X}$. Any such mapping can be decomposed into an orthogonal projection of $\mathbf{X}$ on the subspace followed by a linear mapping within the subspace. The family of distributions obtained by all possible inhomogeneous linear mappings of the uniform distribution on the n-dimensional unit sphere on any subspace of dimension $m \leq n$ may suitably be called the family of ellipsoid distributions of order n. Just as the family of Gaussian distributions of dimension $m \leq n$, the family of ellipsoid distributions of order n is closed with respect to inhomogeneous linear mappings of rank at most equal to n.

Realizations of the random vector $\mathbf{Y}$ are generated by first generating a realization of a standard Gaussian vector $\mathbf{U}$ with unit covariance matrix. Then $\mathbf{X}$ is obtained from $\mathbf{X} = \mathbf{U}/||\mathbf{U}||$. A matrix $\mathbf{A}$ that gives $\mathbf{Y} = \mathbf{AX}$ is determined as any solution to $\mathbf{AA}' = \mathrm{corr}[\mathbf{Y},\mathbf{Y}']$. Finally $\mathbf{Z} = \mathbf{Y}\sqrt{n}$ is a zero mean, unit variance random vector with correlation matrix $\mathrm{corr}[\mathbf{Y},\mathbf{Y}']$. The range of each of the components of $\mathbf{Z}$ is from $-\sqrt{n}$ to $\sqrt{n}$. By multiplication by $\sigma > 0$ and by adding $\mu > \sigma\sqrt{n}$, a random vector of positive components is obtained. It is seen that the coefficient of variation V of the components of this random vector becomes bounded by $1/\sqrt{n}$. Fig. 5 shows examples of translated realizations of $\mathbf{Z}$ with $\mathbf{A}$ computed from both a differentiable and a non-differentiable correlation function.

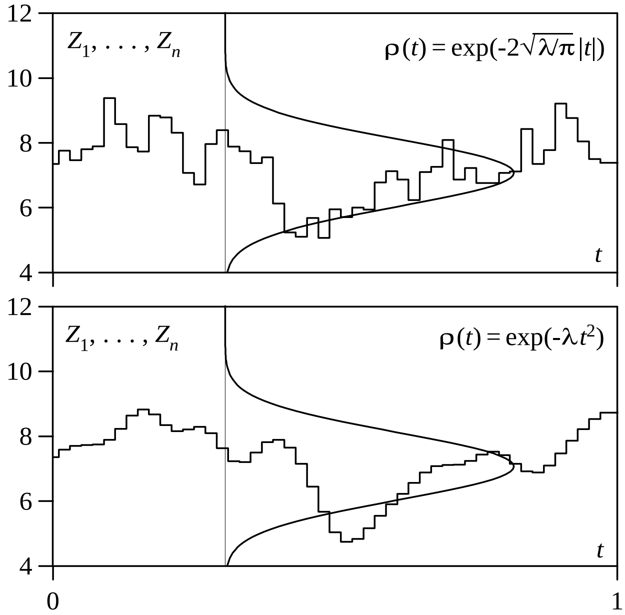

Figure 5: Samples of the ellipsoid distributed vector $\mathbf{Z}+\mu\mathbf{e}$ with $\sigma = 1$ and $\mu = \sqrt{n}$ for $n = 50$ and for two different correlation functions with the same correlation length $\int_{-\infty}^{\infty}\rho(t)dt = \sqrt{\pi/\lambda} \approx 0.14$. The shape of the density function is indicated.

APPENDIX: COVARIANCE FOR INVERSE GAUSSIAN FIELD

Let $X(t) = W(t;0)$ and $Y(t) = W(t;\xi)$ and assume that $X(t_0) = x_0$ and $Y(t_0) = y_0$, where $(x_0,y_0) \in \mathbb{R}^2_+$. With the transition probability that the particle at $[X(t),Y(t)]$ during the time increment τ moves from the position (x_0,y_0) to a position in the infinitesimal set $[x,x+dx]\times[y,y+dy]$ being written as $Q_\tau(x_0,y_0,dx,dy)$, the Chapman-Kolmogorov equation reads (see Fig. 6)

$$p(s+\tau,t+\tau,x_1,y_1;x_0,y_0)$$
$$= \int_x\int_y Q_\tau(x_0,y_0,dx,dy)p(s,t,x_1,y_1;x,y) \tag{34}$$

where $p(s,t,x_1,y_1;x,y)$ is the probability density that $X(s) = x_1, Y(t) = y_1$ given that $[X(0),Y(0)]$ = (x,y).

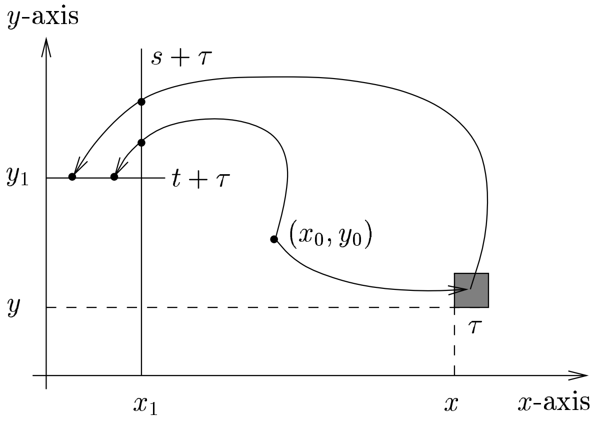

Figure 6: Illustration of transition paths.

By second order Taylor expansion of $p(s,t,x_1,y_1;x,y)$ in (34) with respect to (x,y) from the point (x_0,y_0) and using that $\int_x\int_y (x-x_0)^\alpha (y-y_0)^\beta Q_\tau(x_0,y_0,dx,dy)/\tau \to$ -1 for $(\alpha,\beta)=(1,0)$ or $(0,1)$, σ^2 for $(\alpha,\beta)=(2,0)$ or $(0,2)$, $\rho\sigma^2$ for $(\alpha,\beta)=(1,1)$, and 0 for $(\alpha,\beta)>(2,2)$ as $\tau\to 0$, it follows that p satisfies the diffusion equation

$$\frac{\partial p}{\partial \tau} = -\Big(\frac{\partial p}{\partial x_0}+\frac{\partial p}{\partial y_0}\Big) + \tfrac{1}{2}\sigma^2\Big(\frac{\partial^2 p}{\partial x_0^2}+2\rho\frac{\partial^2 p}{\partial x_0\partial y_0}+\frac{\partial^2 p}{\partial y_0^2}\Big) \qquad (35)$$

Obviously the integral

$$P(s,t;x_0,y_0) = \int_{x_1>0}\int_{y_1>0} p(s,t,x_1,y_1;x_0,y_0)\,dx_1dy_1 \qquad (36)$$

is the probability that $X(s)>0, Y(t)>0$ given that $[X(0),\,Y(0)]=(x_0,y_0)$. Both $P(s,t;x_0,y_0)$ and $q(s,t;x_0,y_0)=\partial^2 P(s,t;x_0,y_0)/\partial s\partial t$ satisfy (35). If the boundary conditions $P(s,t;0,y_0)=P(s,t;x_0,0)=0$, all (s,t), are imposed (absorbing boundaries), the implication is that $P(s,t;x_0,y_0)$ becomes the probability of the event $X(s_1)>0$, $Y(t_1)>0$ for all $s_1\le s, t_1\le t$, which is the same as the event $T_1>s, T_2>t$ where $T_1=T(0;x_0)$ and $T_2=T(\xi;y_0)$ [using self-evident notation noting that $T(0;1)=T(0)$ and $T(\xi;1)=T(\xi)$]. From this it follows directly that the moment generating function

$$M(\theta_1,\theta_2;x_0,y_0)=E[\exp(-\theta_1T_1-\theta_2T_2)] = \int_0^\infty\int_0^\infty e^{-\theta_1 s-\theta_2 t}q(s,t;x_0,y_0)\,dsdt \qquad (37)$$

satisfies the partial differential equation (35) with the left hand side replaced by $(\theta_1+\theta_2)M$. Since $E[T_1T_2]=[\partial^2M/\partial\theta_1\partial\theta_2]_{(0,0)}$ and $[\partial^2[(\theta_1+\theta_2)M]/\partial\theta_1\partial\theta_2]_{(0,0)}=[\partial M/\partial\theta_1+\partial M/\partial\theta_2]_{(0,0)}=-E[T_1]-E[T_2]=-x_0-y_0$ it follows that the function $\mu(x_0,y_0)=E[T_1T_2]$ satisfies the partial differential equation (35) with the left hand side replaced by $-x_0-y_0$. It is directly verified that $\mu_1(x_0,y_0)=x_0y_0+\frac{1}{2}\rho\sigma^2(x_0+y_0)$ is a particular solution. However, this solution does not satisfy the boundary conditions $\mu(x_0,0)=\mu(0,y_0)=0$. We therefore look for a solution μ_2 to the homogeneous equation such that $\mu_1(x_0,y_0)+\mu_2(x_0,y_0)$ will satisfy these boundary conditions.

Applying the linear transformation

$$u=a(x_0+y_0),\quad v=b(x_0-y_0) \qquad (38)$$

where $a=\sigma^{-1}(1+\rho)^{-1/2}$ and $b=\sigma^{-1}(1-\rho)^{-1/2}$, and defining $\psi(u,v)=\exp(-au)\mu(x_0,y_0)$ where $\mu(x_0,y_0)$ satisfies the homogeneous equation, it is seen that ψ satisfies

$$\frac{\partial^2\psi}{\partial u^2}+\frac{\partial^2\psi}{\partial v^2}-a^2\psi=0 \qquad (39)$$

Moreover, ψ must satisfy the boundary conditions

$$\psi(ax_0,bx_0)=-\tfrac{1}{2}\rho\sigma^2x_0e^{-a^2x_0}$$
$$\psi(ay_0,-by_0)=-\tfrac{1}{2}\rho\sigma^2y_0e^{-a^2y_0} \qquad (40)$$

In the polar coordinates (r,ϕ) defined by $u=r\cos\phi$, $v=r\sin\phi$ the equation (39) reads

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}-a^2\psi=0 \qquad (41)$$

We can now proceed to separate variables. Let $\psi=F(r)G(\phi)$. Then, to ensure periodic particular solutions with respect to ϕ we must put $G''=-\nu^2G$ where ν is any number. It follows therefore that

$$\frac{d^2F}{dr^2}+\frac{1}{r}\frac{dF}{dr}-(a^2+\frac{\nu^2}{r^2})F=0 \qquad (42)$$

All solutions to this differential equation, bounded as $r\to 0$, are proportional to $I_\nu(ar)$, where I_ν is the νth order modified Bessel function of the first kind. We are only interested in a bounded solution as $r\to 0$, and furthermore we are only interested in a solution which is even in ϕ. The boundary conditions (40) can jointly be written as

$$\psi(u,\frac{b}{a}u)=\psi(u,-\frac{b}{a}u)=\frac{\rho\sigma^2}{2ab}(-ar\,e^{-ar\cos\phi_0}\sin\phi_0) = \frac{\rho\sigma^2}{ab}\sum_{n=1}^{\infty}(-1)^n nI_n(ar)\sin n\phi_0 \qquad (43)$$

where $\phi_0=\arctan(b/a)$, and the expansion on the right side is obtained by differentiation of the expansion, [1] 9.6.34, p.376,

$$\exp(z\cos\theta)=I_0(z)+2\sum_{n=1}^{\infty}I_n(z)\cos(n\theta) \qquad (44)$$

with respect to θ. This boundary condition is satisfied if we write the solution on the form

$$\psi(u,v) = \frac{\rho\sigma^2}{ab}\Big[\sum_{n=1}^{\infty}(-1)^n n \tan n\phi_0 \, I_n(ar)\cos n\phi - \sum_{m=1}^{\infty} A_{\nu_m} I_{\nu_m}(ar)\cos\nu_m\phi\Big] \qquad (45)$$

$$\nu_m = \frac{(2m-1)\pi}{2\phi_0} \qquad (46)$$

where $A_{\nu_1}, \ldots, A_{\nu_m}, \ldots$ is a sequence of coefficients. This sequence is determined by a condition of correct behaviour at infinity; asymptotically the solution $\mu_2(x_0, y_0)$ must not increase faster than linear in x_0 and y_0 as $x_0 \to \infty$ and $y_0 \to \infty$. Thus we must have that $\psi(u,v) \to 0$ as $r \to \infty$. Since $I_n(ar) \sim I_{\nu(n)}(ar)$ as $r \to \infty$, [1] 9.7.1, p.377, it follows from (45) that A_{ν_m} must be put to the Fourier coefficient

$$A_{\nu_m} = \sum_{n=1}^{\infty}(-1)^n n \tan n\phi_0 B_{\nu_m} \qquad (47)$$

$$B_{\nu_m} = \frac{2}{\phi_0}\int_0^{\phi_0} \cos n\phi \cos\nu_m\phi \, d\phi = \frac{\sin(n+\nu_m)\phi_0}{(n+\nu_m)\phi_0} + \frac{\sin(n-\nu_m)\phi_0}{(n-\nu_m)\phi_0} \qquad (48)$$

Thus the solution can be written

$$\psi(u,v) = \frac{\rho\sigma^2}{ab}\Big\{\sum_{n=1}^{\infty}(-1)^n n \tan n\phi_0 \cdot \Big[I_n(ar)\cos n\phi - \sum_{m=1}^{\infty} B_{\nu_m} I_{\nu_m}(ar)\cos\nu_m\phi\Big]\Big\} \qquad (49)$$

Transforming back to $\mu(x_0, y_0)$, setting $(x_0, y_0) = (1, 1)$, and dividing by σ^2, we finally obtain the correlation coefficient $\mathrm{corr}[T_1, T_2] = [\mu_1(1,1) + \mu_2(1,1) - 1]/\sigma^2$ as given in (27).

ACKNOWLEDGMENTS

This work was initiated when A.M.Hasofer in the summer of 1996 was a visiting professor at the Department of Structural Engineering and Materials at the Technical University of Denmark. The work has been financially supported by the Danish Technical Research Council.

REFERENCES

[1] Abramowitz, M. and Stegun, I.A.: *Handbook of mathematical functions* (ISBN 0-486-61272-4), Dover Publications, New York, 1972

[2] Ditlevsen, O. (1996): "Dimension reduction and discretization in stochastic problems by regression method" in Casciati, F. and Roberts, B. (eds.): *Mathematical Models for Structural Reliability Analysis*, CRC Mathematical Modelling Series (ISBN 0-8493-9631-X), 1996, 51-138.

[3] Ditlevsen, O., Mohr, G., and Hoffmeyer, P.(1996): "Integration of non-Gaussian fields". *Probabilistic Engineering Mechanics*, **11**(1),15-23.

[4] Ditlevsen, O., and Madsen, H.O.: *Structural Reliability Methods*, (ISBN 0-471-96086-1), Wiley, Chichester, 1996.

[5] Ditlevsen, O. and Tarp-Johansen, N.J.: Choice of input fields in stochastic finite elements. Keynote lecture. *Proc. of Seventh Specialty Conference on Probabilistic Mechanics & Structural Reliability*, Worcester, Massachusetts, (eds.: D.M. Frangopol and M.D. Grigoriu), ASCE, Aug. 1996. 820-837.

[6] Ren, Y.J., Elishakoff, I., and Shinozuka, M. (1996): Finite element method for stochastic beams based on variational principles. *Journal of Applied Mechanics*, ASME, (JAM Paper No. 95133).

[7] Mohr, G., Higher moments of weighted integrals of non-Gaussian fields *Proc. of Seventh Specialty Conference on Probabilistic Mechanics & Structural Reliability*, Worcester, Massachusetts, (eds.: D.M. Frangopol and M.D. Grigoriu), ASCE, Aug. 1996. 254-257.

[8] Johnson, L.M., and Kotz, S.: *Distributions in statistics: Continuous univariate distributions-1*, (ISBN 0-471-446260-2), Wiley, New York, 1970.

[9] Johnson, L.M., and Kotz, S.: *Distributions in statistics: continuous multivariate distributions*, (ISBN 0-471-44370-0), Wiley, New York, 1972.

[10] Winterstein, S.R., Nonlinear vibration models for extremes and fatigue, *Journal of Engineering Mechanics*, ASCE, **114**, 1988, 1772-1790.