Slepian simulation of plastic displacement distributions for shear frame excited by filtered Gaussian white noise ground motion

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ABSTRACT: Application of the Slepian model process concept to obtain approximate plastic displacement distributions of elasto-plastic shear frame oscillators of one or more degrees of freedom has in previous works been for white noise force excitation acting directly on the first floor mass of the shear frame. A suitable number of the lower floors has been considered to represent the soil both as a filter of a white noise base rock excitation and as a simplified model for soil structure interaction. In the present paper the Slepian model is applied to obtain plastic displacement distributions for a single story shear frame excited by stationary Gaussian ground motion defined by the output of a Clough-Penzien filter with Gaussian white noise input. This is equivalent to considering an artificial three story elasto-plastic shear frame with possible yielding solely in the third column connection, and with no force feed back from any of the three masses to masses situated below it. The first mass from the ground is force excited by stationary Gaussian white noise as specified in the Clough-Penzien filter definition. The study includes comparisons between plastic displacement results for the same three story frame with partial or full feed back from the movement of the top mass to the second and the first mass (top soil layer mass and base rock mass, respectively).

1 INTRODUCTION

This paper deals with the effect of combining linear filtering of Gaussian white noise ground excitation and soil structure interaction for elasto-plastic shear frames. The model is obtained by a particular interpolation between the well-known Clough-Penzien filter and the MDOF elastic shear frame filter used in several previous works of the authors on the MDOF elasto-plastic oscillator. The Clough-Penzien filter with stationary white noise input gives a stationary ground motion with a particular spectral shape. The MDOF elastic shear frame filter lets the ground motion be modeled as the motion of the n_1th floor in an $n = n_1 + n_2$ floor shear frame where the connections between the n_1 first floor masses are linear elastic and are subject to linear viscous damping. The motion is generated by a Gaussian white noise force excitation acting on the first floor mass. Since there is a feed back from the motion of the top n_2 floor masses to the bottom n_1 floor masses this shear frame filter may at the same time be considered as a model for soil structure interaction between the structure (the n_2 top floors) and the soil (the n_1 bottom floors).

The Clough-Penzien filter is, in fact, a viscously damped elastic shear frame filter with $n_1 = 2$ but with complete cut off of the feed back from the second floor to the first floor and from the third floor to the second floor. The filter considered herein is defined by letting the cut off of feed back be only partly ranging from no cut off to full cut off. Herein the investigation is limited to the case $n_1 = 2$ and $n_2 = 1$.

Let w be the Gaussian white noise applied solely to the first floor of the shear frame. For $\nu_1 = \nu_2 = 1$ and $y_1 = x_1$, $y_2 = x_1 + x_2$, $y_3 = x_1 + x_2 + x_3$, the three differential equations

$$m_1\ddot{y}_1 - \nu_1 c_2 \dot{x}_2 + c_1 \dot{x}_1 - \nu_1 k_2 x_2 + k_1 x_1 = w$$

$$m_2\ddot{y}_2 - \nu_2 c_3 \dot{x}_3 + c_2 \dot{x}_2 - \nu_2 k_3 x_3 + k_2 x_2 = 0$$

$$m_3\ddot{y}_3 + c_3 \dot{x}_3 + k_3 x_3 = 0 \tag{1}$$

obviously describe the displacements y_1, y_2, y_3 of an idealized three-story shear frame with concentrated floor masses m_1, m_2, m_3 (counted from the ground), linear elastic spring stiffnesses k_1, k_2, k_3 between adjacent floors, and linear viscous damping coefficients c_1, c_2, c_3 between adjacent floors.

If $\nu_1 = \nu_2 = 0$ the first two equations of (1) define y_2 as a filtered white noise of Clough-Penzien type (Clough & Penzien 1975). The physical interpretation of this filter is seen from (1) for $\nu_1 = \nu_2 = 1$. By letting m_1, c_1, k_1 be much larger than m_2, c_2, k_2, and m_2, c_2, k_2 be much larger than m_3, c_3, k_3, the terms $-c_2\dot{x}_2$ and $k_2 x_2$ in the first equation and the terms $-c_3\dot{x}_3$ and $k_3 x_3$ in the second equation can be neglected. This situation corresponds to the assumption that the response y_1 to the white noise input models the base rock movement from an earthquake, and that this movement is not influenced by feed back from the movement of the soil layer between the base rock and the ground surface (the foundation level). The movement of the ground surface is supposed to be modeled by the response y_2 of the oscillator governed by the second differential equation excited by the movement of the base rock.

The widely used Kanai-Tajimi filter (Kanai 1957, Tajimi 1960) is obtained if solely the second differential equation is used with the base rock acceleration $\ddot{y}_1$ modeled as stationary white noise. This implies that $c_2\dot{y}_1 + k_2 y_1$ becomes a linear combination of a Wiener process (Brownian motion process) and its integral. Such an excitation process is nonstationary causing the surface to drift in time with increasing variance. This inconvenience is removed by the Clough-Penzien filter.

The purpose of the present investigation is to see the effect of different degrees of feed back to the filter on the process of plastic displacements of the third floor when it is assumed the connection between second and third floor possesses linear-elastic-ideal-plastic behavior. The degree of feed back is defined by the choice of the parameters ν_1 and ν_2. Moreover the purpose is to demonstrate that the method of Slepian model simulation presented in (Lazarov & Ditlevsen 2003) applies also for filtered Gaussian white noise excitation. Herein the investigation is limited to $n_1 = 2$ and $n_1 = 1$.

The behavior of the one degree of freedom elasto-plastic shear frame oscillator (EPO) placed on top of the two degrees of freedom filter shear frame is derived by the aid of the so-called associate linear oscillator (ALO) obtained from the EPO by moving the yield limit to infinity. The key to the solution is that the stationary joint response of the ALO and the linear elastic filter is a Gaussian vector process. In the vicinity of an ALO response crossing to the plastic domain with given velocity the component sample functions can be modeled by the conditional mean vector (linear regression) plus a Gaussian non-stationary residual process. Since the ALO crossing level is given and the velocity of the ALO response at the crossing has a Rayleigh distribution with known mean, unconditioning defines the sample curves in terms of a Rayleigh random variable and the Gaussian residual process. This is the so-called Slepian model associated to the outcrossing.

The symmetric yield limit of the EPO is assumed to be high as compared to the standard deviation of the stationary ALO response. This is to ensure that the outcrossings to the plastic domain can be characterized as occurring in clumps with long time intervals between the clumps. The random behavior of a clump is then approximately independent of the random behavior of any other clump. The rare occurrences of the clumps make it a reasonable assumption that the waiting time from the start of a clump to the start of the next clump gets an exponential distribution.

The start of a clump is conditioned on the event that the response of the EPO is in the elastic domain for at least a time which is longer than or equal to the longest period of the joint response of the ALO and the filter system. As in (Ditlevsen & Bognár 1993) the end of a clump is per definition at the time of the first EPO response return to the elastic domain with a stay there at least through one EPO response extreme.

The first plastic displacement Δ_1 in a clump of plastic displacements with alternating sign has a different distribution than that of the second and the following displacements $\Delta_2, \Delta_3, \ldots$. This is due to a change of the initial conditions. The initial conditions for the ALO at the end of any plastic displacement movement correspond to start with zero velocity at a displacement equal to the relevant yield limit displacement of the EPO. However, for the first plastic displacement in a clump the condition is that the ALO response has been within the yield limit displacement of the EPO at least for one free ALO response period before the occurrence of a local response extreme outside the elastic interval for the EPO. Thus the first plastic displacement is smaller in the mean than the second and the following plastic displacements in a clump.

The distributions of the single plastic displacements and the clump sizes obtained by fast simulation on the basis of Slepian modeling, are compared to the distributions obtained by time consuming direct numerical simulation of the response of the EPO.

It is finally demonstrated how the described Slepian simulation method can be used to determine distribution functions that approximately bound the plastic displacement distribution as it results from a realistic non-stationary earthquake excitation.

2 DIMENSIONLESS REPRESENTATION

Reformulated to dimensionless form the three differential equations in (1) may be collected in the matrix differential equation

$$\ddot{\mathbf{X}}+\mathbf{C}\dot{\mathbf{X}}+\mathbf{K}\mathbf{X} = \frac{w}{m_1\omega_3^2(1-\zeta_3^2)D[x_3(t)]}(\mathbf{e}_1-\mathbf{e}_2) \qquad (2)$$

where $\mathbf{X}(\tau) = [x_1(t)\ x_2(t)\ x_3(t)]^{\mathsf{T}}/D[x_3(t)]$, $\mathbf{e}_1 = [1\ 0\ 0]^{\mathsf{T}}$, $\mathbf{e}_2 = [0\ 1\ 0]^{\mathsf{T}}$, and with $\mu_1 = m_3/m_1$, $\mu_2 =$

m_3/m_2, $\gamma_1 = c_1/c_3$, $\gamma_2 = c_2/c_3$: $\mathbf{C} =$

$$\frac{c_3}{m_3\omega_3\sqrt{1-\zeta_3^2}}\begin{bmatrix} \gamma_1\mu_1 & -\nu_1\gamma_2\mu_1 & 0 \\ -\gamma_1\mu_1 & \gamma_2(\mu_2+\nu_1\mu_1) & -\nu_2\mu_2 \\ 0 & -\gamma_2\mu_2 & 1+\nu_2\mu_2 \end{bmatrix} \quad (3)$$

$\mathbf{K}$ is obtained from $\mathbf{C}$ by squaring $\omega_3\sqrt{1-\zeta_3^2}$, replacing c_3 by k_3, and γ_1, γ_2 by $\kappa_1 = k_1/k_3$, $\kappa_2 = k_2/k_3$. The dimensionless time is $\tau = \omega_3\sqrt{1-\zeta_3^2}\,t$, $\omega_3 = \sqrt{k_3/m_3}$, $2\zeta_3\omega_3 = c_3/m_3$. The dimensionless yield limit is $u = x_{\text{yield }3}/D[x_3(t)]$.

Let $\mathbf{Z}^\mathsf{T} = [\mathbf{X}^\mathsf{T}\ \dot{\mathbf{X}}^\mathsf{T}]$. Then (2) is tantamount to the first order matrix differential equation

$$\dot{\mathbf{Z}} = \mathbf{A}\mathbf{Z} + W(\mathbf{e}_4 - \mathbf{e}_5) \quad (4)$$

where $\mathbf{e}_4 = [0\,0\,0\,1\,0\,0]^\mathsf{T}$, $\mathbf{e}_5 = [0\,0\,0\,0\,1\,0]^\mathsf{T}$, and

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & -\mathbf{C} \end{bmatrix} \quad (5)$$

and where $W(\tau) = w(t)/(m_1\omega_3^2(1-\zeta_3^2)D[x_3(t)])$ is white noise of intensity $\pi S_W = \pi S_w/\{m_1^2\omega_3^3(1-\zeta_3^2)^{3/2}\mathrm{Var}[x_3(t)]\}$. The solution to (5) is

$$\mathbf{Z}(\tau) = e^{\mathbf{A}\tau}\mathbf{Z}(0) + \int_0^\tau e^{\mathbf{A}(\tau-s)}W(s)\,\mathrm{d}s\,(\mathbf{e}_4 - \mathbf{e}_5) \quad (6)$$

The mean of the stationary solution is zero and the covariance matrix function is for $\tau \geq 0$:

$$\mathrm{Cov}[\mathbf{Z}(0), \mathbf{Z}(\tau)^\mathsf{T}] = \mathrm{Cov}[\mathbf{Z}(0), \mathbf{Z}(0)^\mathsf{T}]\,e^{\mathbf{A}^\mathsf{T}\tau} \quad (7)$$

where $\mathrm{Cov}[\mathbf{Z}(0), \mathbf{Z}(0)^\mathsf{T}] = \mathrm{Cov}[\mathbf{Z}(\tau), \mathbf{Z}(\tau)^\mathsf{T}]$ is

$$\mathrm{Cov}[\mathbf{Z}(0), \mathbf{Z}(0)^\mathsf{T}] = e^{\mathbf{A}\tau}\mathrm{Cov}[\mathbf{Z}(0), \mathbf{Z}(0)^\mathsf{T}]\,e^{\mathbf{A}^\mathsf{T}\tau} + \int_0^\tau\!\!\int_0^\tau e^{\mathbf{A}(\tau-s_1)}\mathbf{E}\,e^{\mathbf{A}^\mathsf{T}(\tau-s_2)}\mathrm{Cov}[W(s_1), W(s_2)]\,\mathrm{d}s_1\,\mathrm{d}s_2 \quad (8)$$

in which $\mathbf{E} = (\mathbf{e}_4 - \mathbf{e}_5)(\mathbf{e}_4 - \mathbf{e}_5)^\mathsf{T}$. The white noise property $\mathrm{Cov}[W(s_1), W(s_2)] = \pi S_W\delta(s_1 - s_2)$ implies that the double integral reduces to $\pi S_W\int_0^\tau e^{\mathbf{A}s}\mathbf{E}\,e^{\mathbf{A}^\mathsf{T}s}\,\mathrm{d}s$. Since $e^{\mathbf{A}\tau}$ vanishes as $\tau \to \infty$ it follows from (8) that

$$\mathrm{Cov}[\mathbf{Z}(0), \mathbf{Z}(0)^\mathsf{T}] = \pi S_W\int_0^\infty e^{\mathbf{A}s}\mathbf{E}\,e^{\mathbf{A}^\mathsf{T}s}\,\mathrm{d}s \quad (9)$$

Assume that $\mathbf{A}$ has 6 linearly independent eigenvectors collected as columns in the matrix $\mathbf{V}$ with the corresponding eigenvalues $\lambda_1, \ldots, \lambda_6$ in the diagonal matrix $\mathbf{L}$. Then $e^{\mathbf{A}s} = \mathbf{V}e^{\mathbf{L}s}\mathbf{V}^{-1} = \mathbf{V}^*e^{\mathbf{L}^*s}(\mathbf{V}^{-1})^*$ (* is complex conjugate) where $e^{\mathbf{L}s} = \lceil e^{\lambda_1 s}\cdots e^{\lambda_6 s}\rfloor$. Thus (8) becomes

$$\mathrm{Cov}[\mathbf{Z}(0), \mathbf{Z}(0)^\mathsf{T}]$$

$$= \pi S_W\mathbf{V}\int_0^\infty e^{\mathbf{L}s}\mathbf{V}^{-1}\mathbf{E}\,(\mathbf{V}^{-1})^{\mathsf{T}*}e^{\mathbf{L}^*s}\,\mathrm{d}s\,\mathbf{V}^{\mathsf{T}*}$$

$$= \pi S_W\mathbf{V}\left\{\frac{b_{ij}}{\lambda_i + \lambda_j^*}\right\}_{(6,6)}\mathbf{V}^{\mathsf{T}*} \quad (10)$$

where b_{ij} is the element in the ith row and the jth column of the matrix $\mathbf{B} = \mathbf{V}^{-1}\mathbf{E}(\mathbf{V}^{-1})^{\mathsf{T}*}$.

3 SLEPIAN SIMULATION

Instead of simulating the time history of the total response of the EPO by use of a numerical integration method to obtain the accumulated plastic displacements, it is possible to make a much faster approximate evaluation of the plastic response by use of a so-called Slepian model vector process. Focus is then solely on the interesting events in the response, that is, the crossing events and the following plastic displacements in a clump.

3.1 *Slepian model vector process*

The vector process obtained by conditioning the stationary Gaussian vector process $\mathbf{Z}(\tau)$ on $X_3(0) = Z_3(0)$ and $\dot{X}_3(0) = Z_6(0)$ is also Gaussian, and it can be written as $[\mathbf{Z}(\tau)|Z_3(0), Z_6(0)]$

$$= E[\mathbf{Z}(\tau)|Z_3(0), Z_6(0)] + \mathbf{R}(\tau) \quad (11)$$

The first term is the conditional mean vector function

$$E[\,\mathbf{Z}(\tau)\mid Z_3(0), Z_6(0)] = \frac{\mathrm{Cov}[\mathbf{Z}(\tau), Z_3(0)]}{\mathrm{Var}[Z_3(0)]}Z_3(0) + \frac{\mathrm{Cov}[\mathbf{Z}(\tau), Z_6(0)]}{\mathrm{Var}[Z_6(0)]}Z_6(0) \quad (12)$$

using that $\mathrm{Cov}[Z_3(0), Z_6(0)] = 0$, and the second term is the residual vector process $\mathbf{R}(\tau)$, which is a zero mean Gaussian vector process with covariance matrix function

$$\mathrm{Cov}[\mathbf{R}(\tau_1), \mathbf{R}^\mathsf{T}(\tau_2)] = \mathrm{Cov}[\,\mathbf{Z}(\tau_1), \mathbf{Z}^\mathsf{T}(\tau_2)] - \frac{\mathrm{Cov}[\mathbf{Z}(\tau), Z_3(0)]^2}{\mathrm{Var}[Z_3(0)]} + \frac{\mathrm{Cov}[\mathbf{Z}(\tau), Z_6(0)]^2}{\mathrm{Var}[Z_6(0)]} \quad (13)$$

The variances $\mathrm{Var}[Z_3(0)]$ and $\mathrm{Var}[Z_6(0)]$ are the 0^{th} and the 2^{nd} spectral moments λ_0 and λ_2, respectively, of the scalar process $X_3(\tau)$.

Under the assumption that the vector process $\mathbf{Z}(\tau)$ is stationary, a Slepian model vector process for $\mathbf{Z}(\tau)$ is obtained from (11) by setting the conditioning variables $Z_3(0)$ and $Z_6(0)$ to $Z_3(0) = u$ and $[Z_6(0)\,|\,u$–upcrossing of $Z_3(0)$ at $\tau = 0]$, respectively. The upcrossing is a so-called horizontal window upcrossing. This implies that the last random variable has a Rayleigh distribution with standard deviation $\sqrt{\lambda_2}$. The interpretation is that if the upcrossing velocities $Z_6 = \dot{X}_3$ are recorded at all upcrossings of $Z_3 = X_3$ through level u along a trajectory of $\mathbf{Z}(\tau)$, then the empirical distribution of the upcrossing velocity will approach a Rayleigh distribution as the time of observation of the trajectory increases beyond any limit (Leadbetter, Lindgren, & Rootzén 1983).

3.2 *Simulation of the initial conditions*

The initial conditions at yield limit crossings of the EPO can be simulated by use of this Slepian model vector process. At the time when Z_3 crosses out through the yield limit, a crossing velocity Z_6 is generated from the Rayleigh distribution. Thereafter the displacements and velocities Z_1, Z_2, Z_4, Z_5 are generated from the 4-dimensional normal distribution defined by (12) and (13).

However, the Rayleigh distribution is the distribution of the upcrossing velocity at an arbitrary upcrossing, and not of the upcrossing velocity at the first upcrossing in a clump of upcrossings. The distribution of the first upcrossing velocity in a clump is conditional on the event that Z_3 is in the elastic domain at any time before the time of the upcrossing (assuming the previous clump to be far back in the past). Let the zero point of the time axis be translated to the time of the first crossing event of $Z_3(\tau)$ out through the yield limit u. Then the conditional probability density of the initial conditions for $\mathbf{Z}(\tau)$ follows from Bayes' formula:

$$f_{\mathbf{Z}(0)}[\mathbf{z} \,|\, Z_3(\tau < 0) \in \mathcal{E}, Z_3(0) = u] \propto$$

$$P[Z_3(\tau < 0) \in \mathcal{E} \,|\, \mathbf{Z}(0) = \mathbf{z}, Z_3(0) = u, Z_6(0) = z_6]$$

$$\cdot f_{\mathbf{Z}(0)}(\mathbf{z} \,|\, Z_3(0) = u, Z_6(0) = z_6)\, f_{Z_6(0)}(z_6) \qquad (14)$$

The symbol $\mathcal{E}$ denotes the elastic domain. The random trajectory of the process $Z_3(t)$ for all time $\tau < 0$ is written as $Z_3(\tau < 0)$. The product in the third line of (14) is the density of the initial conditions conditioned solely on the crossing event $Z_3(0) = u$, and it is equal to $f_{\mathbf{Z}_{-3-6}(0)}(\mathbf{z}_{-3-6} \,|\, z_3, z_6)\, f_{Z_6(0)}(z_6)$, in which $\mathbf{Z}_{-3-6}(0)$ is the 4-dimensional subvector obtained from $\mathbf{Z}(0)$ by removing the 3th and the 6th element. The first factor in is the 4-dimensional Gaussian density defined by (12) and (13), and the last factor $f_{Z_6(0)}(z_6)$ is the Rayleigh density with standard deviation $\sqrt{\lambda_2}$.

Due to the probability factor $P[Z_3(\tau < 0) \in \mathcal{E} \,|\, \mathbf{Z}(0) = \mathbf{z}, Z_3(0) = u, Z_6(0) = z_6]$ on the right side of (14) it is not obvious how to generate outcomes from the probability density (14). In stead the principle of weighted outcomes can be used if the probability factor can be calculated for each outcome generated from the density $f_{\mathbf{Z}_{-3-6}(0)}(\mathbf{z}_{-3-6} \,|\, z_3, z_6)\, f_{Z_6(0)}(z_6)$. (Ditlevsen & Randrup-Thomsen 1996) (see remark in (Lazarov & Ditlevsen 2003)).

The next problem is that there is no exact way to calculate the probability $P[Z_3(\tau < 0) \in \mathcal{E} \,|\, \mathbf{Z}(0) = \mathbf{z}, Z_3(0) = u, Z_6(0) = z_6]$. However, as for the one degree of freedom oscillator, numerical investigations show that this probability can be approximated well by considering $X_3(\tau)$ at solely one or some few specific time points $-\tau_1 > -\tau_2 > \ldots > -\tau_q$ immediately before the crossing time point $\tau = 0$, if these are chosen as the time points where $Z_3(\tau)$ has local maximum of the probability of being in the plastic domain. The corresponding vector of displacements $[Z_3(-\tau_1) \ldots Z_3(-\tau_q)]$ is Gaussian with mean vector and covariance matrix determined from the linear regression of $[Z_3(-\tau_1) \ldots Z_3(-\tau_q)]$ on $\mathbf{Z}(0) = \mathbf{z}_0$, and the corresponding residual covariance matrix, respectively. The conditioning vector $\mathbf{z}_0$ is the generated outcome from the density $f_{\mathbf{Z}_{-3-6}(0)}(\mathbf{z}_{-3-6} \,|\, z_3, z_6)\, f_{Z_6(0)}(z_6)$.

According to (7) the ith element of the conditional mean vector is $E[Z_3(-\tau_i)|\mathbf{z}_0] = \mathrm{Cov}[Z_3(0), \mathbf{Z}^\mathsf{T}(0)]e^{\mathbf{A}^\mathsf{T}\tau_i}\mathrm{Cov}[\mathbf{Z}(0), \mathbf{Z}(0)^\mathsf{T}]^{-1}\mathbf{z}_0$, and the conditional covariance between the ith and jth element is $\mathrm{Cov}[Z_3(-\tau_i), Z_3(-\tau_j)|\mathbf{z}_0] = \mathrm{Cov}[Z_3(0), \mathbf{Z}(0)^\mathsf{T}]e^{\mathbf{A}^\mathsf{T}|\tau_i-\tau_j|}\mathbf{e}_3 - \mathrm{Cov}[Z_3(0), \mathbf{Z}(0)^\mathsf{T}]e^{\mathbf{A}^\mathsf{T}\tau_i}\mathrm{Cov}[\mathbf{Z}(0), \mathbf{Z}(0)^\mathsf{T}]^{-1}e^{\mathbf{A}\tau_j}\mathrm{Cov}[\mathbf{Z}(0), Z_3(0)]$, with $\mathrm{Cov}[\mathbf{Z}(0), \mathbf{Z}(0)^\mathsf{T}]$ given by (10). Numerical methods for calculating the probability of the multidimensional Gaussian distribution are discussed in (Genz 1992).

3.3 *First plastic displacement in a clump*

Assuming that the ALO response $Z_3(\tau)$ for a long time has been totally inside the elastic domain, an approximate simulation of the EPO behavior after a u-upcrossing event of $Z_3(\tau)$ at time $\tau = 0$, say, can be made by use of the Slepian model (11) simulating the initial conditions $\mathbf{Z}(0)$ of the ALO from the joint density $f_{\mathbf{Z}_{-3-6}(0)}(\mathbf{z}_{-3-6} \,|\, z_3, z_6)\, f_{Z_6(0)}(z_6)$. After the upcrossing of Z_3 to the plastic domain, the plastic displacement of the third mass of the EPO adds up as long as the velocity of the mass is positive. After some time τ_0 the velocity becomes zero and the plastic displacement accumulation terminates. Thereafter the oscillator returns to the elastic domain implying that, except for the plastic displacement shift, the EPO again behaves like the ALO with initial velocity $Z_6(\tau_0) = 0$ and initial displacement $Z_3(\tau_0) = u$. The remaining initial values $Z_1(\tau_0), Z_2(\tau_0), Z_4(\tau_0), Z_5(\tau_0)$ are as an approximation set to the values obtained from the Slepian model (11) for the ALO response with a simulated outcome of the residual vector $\mathbf{R}(\tau_0)$.

The velocity of the the third mass has a large number of small fluctuations around zero when the response is about to return to the elastic domain. These small fluctuations are partly caused by the white noise excitation and partly by the contributions from the excited higher eigenfrequencies. The determination of the plastic displacement is not affected by these fluctuations if the approximate energy balance principle in (Karnopp & Scharton 1966) is used. According to this principle, the maximal elastic excess energy at the time of crossing to the plastic domain is set equal to the energy dissipated by the plastic work. In the present case of ideal elasto-plastic behavior, the plastic displacement obtained from the Karnopp-Scharton principle is $\Delta_1 = (Z_{3,\max}^2 - u^2)/(2u)$, where $Z_{3,\max}$ is the largest local maximum of the conditional mean part of the Slepian model for the ALO response Z_3 after the upcrossing and before the following crossing back to the elastic domain.

Since, for sufficiently high yield limits, the time spent in the plastic domain is short compared to the maximum period of the structure, the neglect of the contribution from residual process, that is, the neglect of the excitation, causes only a minor error.

3.4 *Second and following displacements in a clump*

Regression method. With the initial conditions given for the ALO at time τ_0, and shifting the time origin to τ_0, an approximation to the next following negative minimum of $Z_3(\tau)$ is simulated from the regression $[\mathbf{Z}(\tau) \mid Z_3(0) = u, Z_6(0) = 0, \mathbf{Z}_{-3-6}(0)] = \mathrm{Cov}[\mathbf{Z}(\tau), \mathbf{Z}(0)^\mathsf{T}]\,\mathrm{Cov}[\mathbf{Z}(0), \mathbf{Z}(0)^\mathsf{T}]^{-1}[Z_1(0) \quad Z_2(0) \quad u \quad Z_4(0) \quad Z_5(0) \quad 0]^\mathsf{T} + \mathbf{R}(\tau)$ where $\mathrm{Cov}[\mathbf{R}(\tau_1), \mathbf{R}(\tau_2)^\mathsf{T}] = \mathrm{Cov}[\mathbf{Z}(\tau_1), \mathbf{Z}(\tau_2)^\mathsf{T}] - \mathrm{Cov}[\mathbf{Z}(\tau_1), \mathbf{Z}(0)^\mathsf{T}]\,\mathrm{Cov}[\mathbf{Z}(0), \mathbf{Z}(0)^\mathsf{T}]^{-1}\,\mathrm{Cov}[\mathbf{Z}(0), \mathbf{Z}(\tau_2)^\mathsf{T}]$

The time point $\tau_{\min}$ of the negative minimum is approximated as the time point of deepest negative minimum of the expectation $E[Z_3(\tau) \mid Z_3(0) = u, Z_6(0) = 0, \mathbf{Z}_{-3-6}(0)]$. The value of $Z_3(\tau_{\min})$ is simulated by adding the third component of a simulated value of the residual vector $\mathbf{R}(\tau_{\min})$ to the mean value. If the obtained ALO response $Z_3(\tau_{\min})$ is below $-u$, the Karnopp-Scharton principle is applied to calculate the corresponding plastic displacement Δ_2 of the EPO. Termination of a clump of plastic displacements of the EPO is defined such that if the value $Z_3(\tau_{\min})$ is above $-u$, then the clump terminates. If the clump does not terminate the simulation is continued as above with obvious use of the symmetry of the oscillator. The initial conditions of Z_1, Z_2, Z_4, Z_5 are those obtained from $[\mathbf{Z}(\tau) \mid Z_3(0) = u, Z_6(0) = 0, \mathbf{Z}_{-3-6}(0)]$ for $\tau = \tau_{\min}$ with the simulated residual $\mathbf{R}(\tau_{\min})$ substituted. After time origin shift and sign shift everything runs as for the simulation of the plastic displacement Δ_2 to obtain Δ_3 or to decide whether the clump terminates.

Amplitude method. By generalization of the definition in (Tarp-Johansen & Ditlevsen 2001) a conditional amplitude process $A(\tau)$ for $Z_3(\tau)$ is defined by $A[\tau \mid Z_3(0) = u, Z_6(0) = 0, \mathbf{Z}_{-3-6}(0)] = -\sqrt{S(\tau)^2 + \dot{S}(\tau)^2/\lambda_2}$, where $S(\tau)$, $\dot{S}(\tau)$ are the 3th and the 6th component of $[\mathbf{Z}(\tau) \mid Z_3(0) = u, Z_6(0) = 0, \mathbf{Z}_{-3-6}(0)]$, and λ_2 is the second spectral moment of the displacement response process $Z_3(\tau)$, that is, λ_2 is the variance of the velocity process $Z_6(\tau)$. In stead of approximating $Z_3(\tau_{\min})$ by $S(\tau_{\min})$ the approximation $A[\tau_{\min} \mid Z_3(0) = u, Z_6(0) = 0, \mathbf{Z}_{-3-6}(0)]$ can be used. This approximation may be an improvement over $S(\tau_{\min})$ because the random drift of the phase by time causes the minimum to occur at a random time slightly different from $\tau_{\min}$ after the previous maximum. Therefore $S(\tau_{\min})$ underevaluates the minimum in absolute value. However, since the conditional amplitude process varies slowly over a reasonably wide neighborhood of $\tau_{\min}$, its value at $\tau_{\min}$ is an applicable approximation.

Moreover the first approximation does not behave asymptotically correctly as $u \to 0$. Due to the addition of a Gaussian residual with standard deviation independent of u to the conditional mean that decreases to zero with u, the probability that $S(\tau_{\min}) < 0$ approaches 1/2. For the amplitude approximation the minimum is negative per definition. This difference between the two methods is analyzed in detail for the one degree of freedom EPO in (Ditlevsen & Bognár 1993).

3.5 *Sample weighting*

The simulated set of plastic displacements $\Delta_{1i}, \Delta_{2i}, \ldots, \Delta_{Ni}$ in the ith independently simulated clump must be assigned the weight $p_i = P[Z_3(\tau < 0) \in \mathcal{E} \mid \mathbf{Z}(0) = \mathbf{z}, Z_3(0) = u, Z_6(0) = z_6]$ approximately calculated as explained in Section 3.2. This means that in stead of assigning the uniform weight $1/n$ to each of the observations in the sample of size n of independently simulated clumps, the non-uniform normalized weight $p_i/(p_1 + \ldots + p_n)$, must be assigned to the observation $\Delta_{1i}, \Delta_{2i}, \ldots, \Delta_{Ni}$.

4 EXAMPLES

According to (Clough & Penzien 1975), p. 614, Kanai has suggested the eigenfrequency $\omega_{\text{base rock}} = \omega_1 = \sqrt{k_1/m_1} = 15.6$ rad/s and the damping ratio $\zeta_{\text{base rock}} = \zeta_1 = c_1/2\sqrt{m_1 k_1} = 0.6$ as being representative of firm soil conditions. In the following examples these values are used together with the values $\omega_{\text{top soil}} = \omega_2 = \sqrt{k_2/m_2} = \omega_1/5$ and $\zeta_{\text{top soil}} = \zeta_2 = c_2/2\sqrt{m_2 k_2} = \zeta_1$ for the filtering of the quake through the top soil layer. The masses are chosen such that $m_1/m_3 = 100$ and $m_2/m_3 = 10$. Due to the normalization to unit variance of the displacement of the third mass relative to the second mass, the absolute values of the masses m_1, m_2, m_3, stiffnesses k_1, k_2, k_3 and damping coefficients c_1, c_2, c_3 do not enter the calculations.

Different parameter values are chosen for the top frame structure. The example values are $\omega_3/\omega_1 = \sqrt{k_3/m_3}/\sqrt{k_1/m_1} = 0.25$ and 0.75, and $\zeta_3 = c_3/2\sqrt{m_3 k_3} = 0.01$. The dimensionless yield limits $u = 1$ and 3, and the cases $(\nu_1, \nu_2) = (0, 0)$ (no feedback), $(0, 1)$ (feed back solely to top soil layer), and$(1, 1)$ (feed back to base rock and top soil layer) are considered.

For comparisons with the closed form distribution functions (44) and (52) in (Ditlevsen & Bognár 1993) for Δ_1 and Δ_2 (SDOF white noise excited EPO) we need the equivalent value of the damping ratio ζ. For this the last equation in (1) is written as

$$\ddot{x}_3+c_3\left(\frac{1}{m_3}+\frac{\nu_2}{m_2}\right)\dot{x}_3+k_3\left(\frac{1}{m_3}+\frac{\nu_2}{m_2}\right)x_3=\frac{c_2\dot{x}_2+k_2x_2}{m_2} \qquad (15)$$

giving $\zeta = c_3/[2\sqrt{k_3\,(1/m_3+\nu_2/m_2)}\,]$.

Fig. 1 corresponds to the no feed back case $(\nu_1,\nu_2) = (0,0)$. It is seen that the general effect of the filter is that smaller plastic displacements are obtained in the mean as compared to those of the "equivalent" SDOF EPO. The difference decreases with the yield level u and increases with ω_3. Moreover, the comparison with the direct simulation PDF shows that the Slepian method gives very accurate results for Δ_1 and less accurate but acceptable results for Δ_2. Surprisingly, except for a single case, the amplitude method does not give an improvement as compared to the regression method. Generally the results are better in the low frequency range of ω_3 than in the high frequency range.

Inspection of Fig. 2 corresponding to the case $(\nu_1,\nu_2) = (0,1)$, that is, full feed back to the soil layer but no feed back to the base rock, shows the same features except that larger plastic displacements are obtained in the mean as compared to those of the "equivalent" SDOF EPO. Moreover it is seen that the effect of the feed back is a substantial increase of the plastic displacements. The case $(\nu_1,\nu_2) = (1,1)$ of full feed back to both the soil layer and to the base rock shows no observable change from the case $(\nu_1,\nu_2) = (0,1)$. The mass and stiffness of the base rock is too large to cause any significant feed back influence from the coupling between the structure movements and the base rock movements.

With respect to clump size Figs. 3 and 4 show that the filter influence is modest and smallest in the low frequency range of ω_3. In the mean the clump size is larger in the high frequency range than in the low frequency range. The geometric distribution $P(N = n) = (1-p)\,p^{n-1}$ with p given as function of u in (53) of (Ditlevsen & Bognár 1993) fits reasonably well in the low frequency range and might even be applicable in the high frequency range.

Fig. 5 displays the dimensionless spectra of the right side of (15). The bottom diagrams correspond to the feed back cases. There is almost no difference between the spectra for the partial feed back case and the full feed back case. The minima of the spectra are coincident with $\omega_{\text{equivalent}} = \sqrt{k_3(1/m_3+\nu_2/m_2)}$. The horizontal lines show the levels of the equivalent white noises (that is, the noises that like the spectra give unit variance of the response of the SDOF ALO). The crossing points shortly to the left of the minima are almost coincident with the points of the damped eigenfrequencies $\omega_{\text{equivalent}}\sqrt{1-\zeta^2}$.

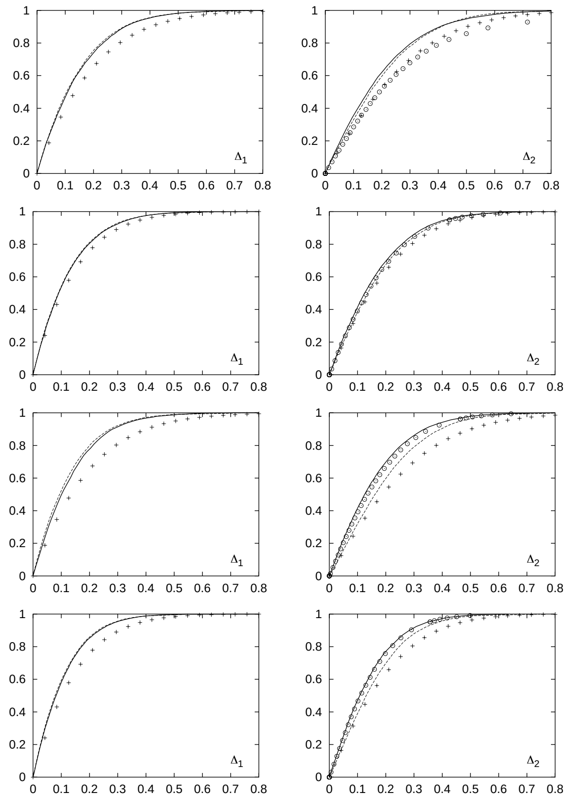

Figure 1. Plastic displacement PDFs (marked by +) (left) and (right) ("equivalent" direct white noise excited SDOF) compared to the PDFs (solid curve) obtained by Slepian simulation (regression method) in the case of Clough-Penzien filtered excitation without feed back $[(\nu_1,\nu_2) = (0,0)]$. The PDFs marked by $\circ$ are obtained by the amplitude method. The PDFs obtained by direct numerical integration simulation are shown as dotted curves. Top to bottom: $\omega_3/\omega_1 = 0.25, 0.25, 0.75, 0.75$, $u = 1, 3, 1, 3$.

5 EARTH QUAKE EXCITATION

For an earthquake similar to the El-Centro earthquake (1940) a suitable stochastic model is obtained by modifying a stationary white noise by multiplication by the deterministic modulating function $\psi(t) = (t/t_1)^2$ for $0 \le t \le 4$, $\psi(t) = 1$ for $4 \le t \le 35$, $\psi(t) = \exp[-0.0357\,(t-35)]$ for $35 \le t \le 80$, and $\psi(t) = 0.05 + 0.938 \times 10^{-4}(120-t)^2$ for $80 \le t \le 120$ s (Jennings, Housner, & Tsai 1969). This modulating function is shown Fig. 6 together with the calculated standard deviation function of the corresponding ALO displacement response of the mass m_3 for the four different cases considered in the previous section. For the case in the upper left diagram, that is, the case of no feed back and $\omega_3/\omega_{\text{base rock}} = 0.25$, the maximal dimensionless standard deviation is 0.96. To reach a maximal standard deviation of 1, the modulated white noise intensity is for that case increased by the factor 1/0.96 in order to make the directly simulated plastic displacement distribution comparable with the distributions obtained from the Slepian method.

The Slepian modeling method can be applied to produce approximate plastic displacement results simply by replacing the nonstationary response by the full stationary response of the unmodulated white

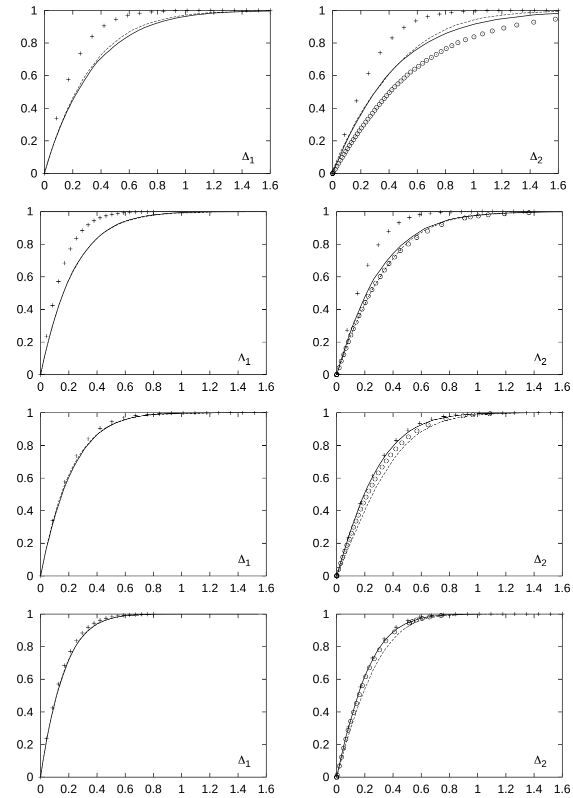

Figure 2. As Fig. 1 except that the Clough-Penzien filter allows feed back to the top soil layer [$(\nu_1, \nu_2) = (0, 1)$] or to both the top soil layer and the base rock [$(\nu_1, \nu_2) = (1, 1)$] (negligible deviation between the two cases). Top to bottom: $\omega_3/\omega_1 = 0.25, 0.25, 0.75, 0.75$, $u = 1, 3, 1, 3$.

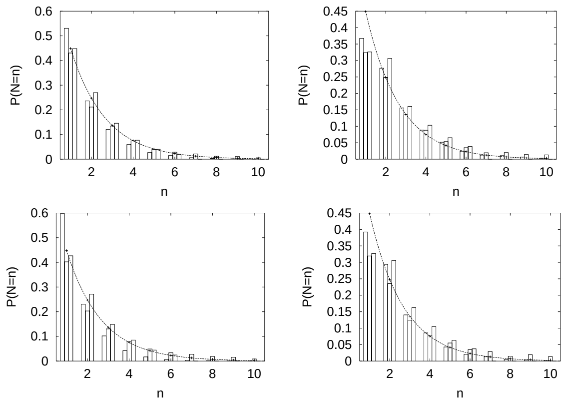

Figure 3. Probability estimates for the clump size N. The curves correspond to the geometric distribution corresponding to the "equivalent" white noise excited SDOF EPO. The columns at $n = 1, 2, \ldots$ correspond to the Slepian method (regression (left) and amplitude (middle)) and to direct numerical integration simulation (right). $u = 1$. Top: no feed back [$(\nu_1, \nu_2) = (0, 0)$]. Bottom: partial or full feed back [$(\nu_1, \nu_2) = (0, 1)$] and [$(\nu_1, \nu_2) = (1, 1)$]. Left: $\omega_3/\omega_1 = 0.25$. Right: $\omega_3/\omega_1 = 0.75$.

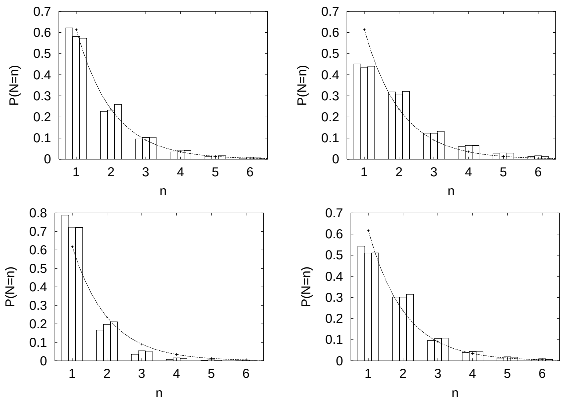

Figure 4. As Fig. 3 except that $u = 3$.

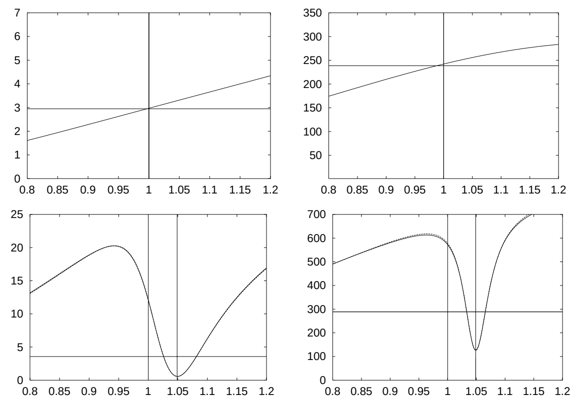

Figure 5. Spectra on dimensionless form of the right side of (15) for $(\nu_1, \nu_2) = (0, 0)$ (top), and $(0, 1), (1, 1)$ (bottom) and for $\omega_3/\omega_{\text{base rock}} = 0.25$ (left) and 0.75 (right). Besides $\omega_3 = \sqrt{k_3/m_3}$ the values of $\omega_{\text{equivalent}} = \sqrt{k_3(1/m_3 + 1/m_2)}$ are marked by vertical lines. The horizontal lines are the equivalent white noise levels.

noise but only considered within a time window determined from the standard deviation function as the largest time interval $[t_1, t_2]$ within which the standard deviation is larger than some suitably chosen level σ_{cut}.

As in (Ditlevsen & Bognár 1993) the distribution of the time from clump end to clump start is approximated by the exponential distribution with parameter κ, that is, with mean value

$$\frac{1}{\kappa} = \frac{\pi}{\sqrt{k_3(1/m_3 + \nu_2/m_2)}} + \sqrt{\frac{2\pi}{\lambda_2 - \lambda_1^2}}\,\frac{e^{u^2/2} - 1}{u} \qquad (16)$$

where λ_n is the nth order one-sided spectral moment of the ALO response. This assessment is obtained by use of the crossing rate of the Hilbert transform envelope of the ALO response combined with the probability $1 - \exp(-u^2/2)$ that at any time point the envelope is below u. This assessment fits well with simulated results.

Let any part of the EPO response that starts at the end of a clump and terminates after the end of the next clump be called a cycle. If the simulated waiting time in the first cycle is larger than $t_2 - t_1$, set the plastic displacements from the earth quake to zero. Otherwise, if the sum of the duration of the first cycle and the waiting time in the second cycles is larger than $t_2 - t_1$, set the plastic displacement from the earth quake to the plastic displacement from the clump of the first cycle. The clump is simulated by first simulating the clump size n from a distribution as in Fig. 3, and next simulating the needed number of outcomes from the distributions of Δ_1 and Δ_2 to obtain the net and the absolute plastic displacement

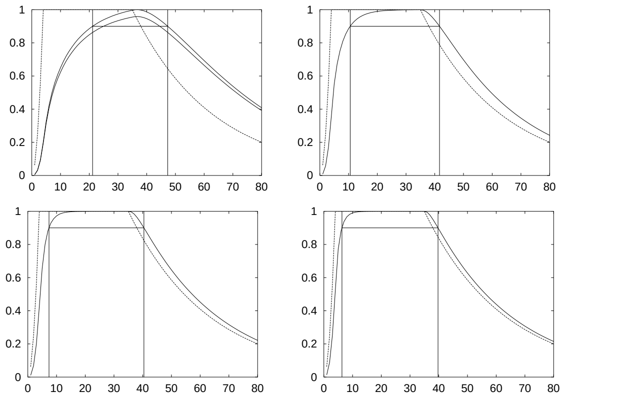

Figure 6. White noise modulation function $\psi(t)$ (thin curve) and corresponding standard deviation function (thick curve). Top: no feed back. Bottom: full feed back. Left: $\omega_3/\omega_1 = 0.25$. Right: $\omega_3/\omega_1 = 0.75$.

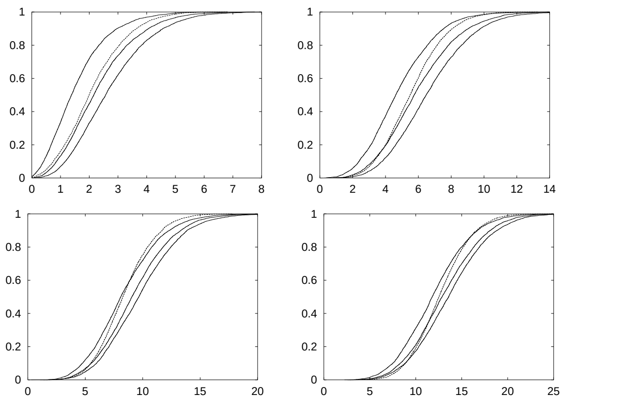

Figure 7. Distribution function approximations for the absolute plastic displacement caused by earthquake simulating filtered Gaussian white noise excitation. Thick curves: Slepian simulation; left to right: $\sigma_{\text{cut}}/\sigma_{\max} = 0.95, 0.90, 0.85$. Thin curve: direct simulation of the nonstationary response. The normalized yield level is $u = 1$. Top: no feed back. Bottom: full feed back. Left: $\omega_3/\omega_1 = 0.25$. Right: $\omega_3/\omega_1 = 0.75$.

$\pm(\delta_1 - \delta_2 + \ldots + (-1)^{n-1}\delta_n$ and $\delta_1 + \delta_2 + \ldots + \delta_n$, respectively, with probability 0.5 on each of the two sign possibilities. If the sum of the durations of the two first cycles and the waiting time in the third cycle is larger than $t_2 - t_1$, set the plastic displacements from the earthquake to the sum of the plastic displacements in the two first cycles, etc.

Fig. 7 shows the obtained distribution functions. It is seen that the standard deviation of the plastic displacement distribution corresponding to the nonstationary directly simulated response is smaller than the standard deviations obtained by the Slepian simulation. Moreover it is seen that the displacement distribution is mainly to the conservative side for $\sigma_{\text{cut}}/\sigma_{\max} = 0.90$, and mostly to the unsafe side when this standard deviation ratio is 0.95.

An important observation is that in the mean the accumulated plastic displacements are about two to three times larger in the feed back case than in the no feed back case.

6 CONCLUSIONS

The Slepian simulation method is well suited for calculating plastic displacement distributions of a one-story shear frame with elasto-ideal-plastic constitutive behavior. The excitation is filtered Gaussian white noise applied at the ground level. The cases range from no feed back to full feed back from the structure to the filter. With respect to earthquake excitations, reasonable results are obtained from the Slepian method by use of an "equivalent" stationary response considered solely in the largest time interval within which the standard deviation of the nonstationary response is larger than about 90 % of the maximal standard deviation. The full feed back case gives the largest plastic displacements.

ACKNOWLEDGEMENT

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