Integrals of Random Fields Treated by the Model Correction Factor Method

Paolo Franchin
Dept. of Civil and Envir. Eng., University of California, Berkeley, CA

Ove Ditlevsen
Dept. of Struct. Eng. and Materials, Technical University of Denmark, Lyngby, Denmark

Armen Der Kiureghian
Dept. of Civil and Envir. Eng., University of California, Berkeley, CA

ABSTRACT: The model correction factor method (MCFM) is used in conjunction with the first-order reliability method (FORM) to solve structural reliability problems involving integrals of non-Gaussian random fields. The approach replaces the limit-state function with an idealized one, in which the integrals are considered to be Gaussian. Conventional FORM analysis yields the linearization point of the idealized limit-state surface. A model correction factor is then introduced to push the idealized limit-state surface onto the actual limit-state surface. A few iterations yield a good approximation of the reliability index for the original problem. This method has application to many civil engineering problems that involve random fields of material properties or loads. An application to reliability analysis of foundation piles illustrates the proposed method.

1 INTRODUCTION

Many problems in structural reliability involve integrals of random fields in the definition of the limit state function. This is usually the case when material properties or distributed load intensities exhibit random spatial variabilities. The issue of integrals over random fields arises in an important way in stochastic finite element analysis (Liu & Der Kiureghian 1991). In such analysis, the coefficients of the element stiffness matrix are obtained in terms of integrals, over the element, of material property fields modulated by appropriate shape functions. Similarly, element nodal loads are obtained as integrals of modulated load intensity fields over the element.

Great advantage is gained if the random fields of interest can be modeled as Gaussian. In that case the distribution of the integral is also Gaussian. Together with easily computable expressions of the mean and variance, full probabilistic characterization of the integral is then possible. However, in many applications the Gaussian assumption is inadmissible because of physical characteristics, e.g., boundedness, of the random field. Unfortunately, the distribution of the integral of a non-Gaussian random field is generally unknown. This problem is the main focus of this paper.

One way to represent the integral of a non-Gaussian random field is to subdivide the domain into small sub-domains and express the integral as the sum of the integrals over the sub-domains (hereafter "sub-integrals"). If each sub-domain is sufficiently small, the marginal distribution of the sub-integral is approximately of the same type as the distribution of the field. Together with easily computed second moments (means, variances and covariances) of the sub-integrals, a Nataf-type joint distribution, see (Liu & Der Kiureghian 1986), for the set of sub-integrals can be constructed. The integral over the entire field is then expressed as the sum of a large number of random variables having a known, Nataf-type joint distribution. This representation can theoretically be used in conjunction with conventional reliability methods such as the first- and second-order reliability methods (FORM and SORM) to solve the problem (Ditlevsen & Madsen 1996). The main issue then is the large number of random variables that result from splitting the integrals into sub-integrals, which could make the required computational effort with conventional reliability methods prohibitively costly.

In the above representation, the random variables representing the sub-integrals are of course correlated. However, if the characteristic correlation scale of the field is small in relation to the size of the integration domain, the correlation would tend to quickly die out with increasing distance between the sub-domains. Then, with a large number of sub-domains, provided the mean and variance of the field are constant or slowly varying, one can invoke the central limit theorem to justify the Gaussian distribution for the sum of the sub-integrals. In fact, this argument has been frequently used in the literature to justify the use of the Gaussian distribution for integrals of non-Gaussian fields. The problem, of course, is that in many real applications the assumptions that justify

the Gaussian distribution are not valid.

In this paper we present a method for FORM analysis that takes advantage of both the approximation methods described above, i.e., representation of the integral in terms of the sum of Nataf-distributed random variables and the Gaussian assumption of the integral. The vehicle is the Model Correction Factor Method (MCFM) previously developed by the second author and his co-workers. Essentially, FORM analysis is carried out by use of an idealized problem that corresponds to the original problem with the integrals assumed to be Gaussian. A model correction factor is then introduced to move the linearization point of the idealized problem onto the limit-state surface of the original problem. The method requires iterative solution of simpler reliability problems with smaller number of random variables than the original problem.

We begin the paper by a brief review of the MCFM. The specific application to reliability problems involving integrals of non-Gaussian random fields is then described. In the last section we present an application of the method to a pile foundation problem. The exercise serves to demonstrate the accuracy and efficiency of the proposed method, as well as the significance of the non-Gaussian distribution of the integrals involved. A more detailed treatment of the problems considered in this paper is given in (Franchin et al. 2000).

2 MODEL CORRECTION FACTOR METHOD

The MCFM has been formulated and tested in several papers (Ditlevsen & Arnbjerg-Nielsen 1994, Johannesen & Ditlevsen 1993, Ditlevsen & Johannesen 1999). A geometric and tutorially sufficient interpretation of the (zeroth order) MCFM is as follows. Let

$$g(\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D) = 0 \tag{1}$$

be the equation of a limit state surface corresponding to some physically interpretable event in the space of $\mathbf{x}_S, \mathbf{x}_R$ and $\mathbf{x}_D$, where $\mathbf{x}_S$ is the vector of input load variables, $\mathbf{x}_R$ is the vector of input resistance variables, and $\mathbf{x}_D$ is the vector of all remaining input variables (of the type as geometrical variables or dimensionless basic variables).

In order for (1) to be dimensionally correct and the division of the input variables into $\mathbf{x}_S, \mathbf{x}_R$ and $\mathbf{x}_D$ make sense, it is necessary that the two vectors $\mathbf{x}_S$ and $\mathbf{x}_R$ share a physical unit. This means that $g(k\mathbf{x}_S, k\mathbf{x}_R, \mathbf{x}_D) = 0$ for any positive k. Thus, for each given $\mathbf{x}_D$, the surface defined by (1) is a cone with its vertex at the origin of $(\mathbf{x}_S, \mathbf{x}_R)$.

In the following we consider limit state equations $g(\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D) = 0$ for which the equation

$$g(k\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D) = 0 \tag{2}$$

has a unique solution for k for any given value of $\mathbf{x} = (\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D)$. If $g_1(\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D) = 0$ and $g_2(\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D) = 0$ define two different cones for any fixed value of $\mathbf{x}_D$, then it is clear from a geometric consideration that cone 2 can be mapped into cone 1 by a unique pointwise affinity $\mathbf{x} = (\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D) \rightarrow (\mathbf{x}_S, \nu(\mathbf{x})\mathbf{x}_R, \mathbf{x}_D)$ such that the equations $g_1(\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D) = 0$ and

$$g_2(\mathbf{x}_S, \nu(\mathbf{x})\mathbf{x}_R, \mathbf{x}_D) = 0 \tag{3}$$

both represent the limit state surface 1. Clearly $\nu(k\mathbf{x}_S, k\mathbf{x}_R, \mathbf{x}_D) = \nu(\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D)$ for any $\mathbf{x}$ and any $k > 0$. In fact, since for any value of $\mathbf{x}$ we have unique values $k_1(\mathbf{x})$ and $k_2(\mathbf{x})$ such that

$$g_1[k_1(\mathbf{x})\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D] = 0 \tag{4}$$

$$g_2[k_2(\mathbf{x})\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D] = 0 \tag{5}$$

which can be replaced by

$$g_2[k_1(\mathbf{x})\mathbf{x}_S, \nu(\mathbf{x})\mathbf{x}_R, \mathbf{x}_D] = 0 \tag{6}$$

$$g_2[k_1(\mathbf{x})\mathbf{x}_S, \frac{k_1(\mathbf{x})}{k_2(\mathbf{x})}\mathbf{x}_R, \mathbf{x}_D] = 0 \tag{7}$$

respectively (the last equation due to the physical dimension homogeneity of g_2), it follows that

$$\nu(\mathbf{x}) = \frac{k_1(\mathbf{x})}{k_2(\mathbf{x})} \tag{8}$$

The idea of the MCFM is to replace an elaborate limit state equation $g_1(\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D) = 0$ by a simpler limit state equation

$$g_2(\mathbf{x}_S, \nu^*\mathbf{x}_R, \mathbf{x}_D) = 0 \tag{9}$$

obtained from (3) by replacing the function $\nu(\mathbf{x})$ by a constant ν^*. This replacement may be applicable in a more or less wide neighborhood of any point $\mathbf{x}^*$ at which $\nu^* = \nu(\mathbf{x}^*)$ is calculated. Thus ν^* acts as a model-correction factor applied to the strength variables in the simpler model 2. For reliability analysis purposes the best choice of the value of this factor obviously is the one that is obtained at the point of linearization in FORM (hereafter the "beta point") of the elaborate limit-state surface. Given that $\nu(\mathbf{x})$ actually has a point of stationarity at $\mathbf{x}^*$, that is, given that all the partial derivatives of $\nu(\mathbf{x})$ are zero at $\mathbf{x}^*$, then the two limit-state surfaces defined by $g_1(\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D) = 0$ and $g_2(\mathbf{x}_S, \nu^*\mathbf{x}_R, \mathbf{x}_D) = 0$, respectively, are tangential to each other at $\mathbf{x}^*$. Thus the two limit-state surfaces have $\mathbf{x}^*$ in common as a point that satisfies the necessary conditions for being a beta point for both surfaces. The search for the point $\mathbf{x}^*$ with these properties may start by applying FORM to (9) with ν^* put to some judgmentally chosen value, say $\nu^* = \nu_1 = 1$. This determines a first approximation $\mathbf{x}_1$ to the beta

point. Equation (4) with $\mathbf{x}_1$ substituted for $\mathbf{x}$ is next solved with respect to $k_1(\mathbf{x}_1)$. The solution is a new value ν_2 of the correction factor because $k_2(\mathbf{x}_1) = 1$. Using $\nu^* = \nu_2$ in (9), FORM analysis gives a new approximation $\mathbf{x}_2$ to the beta point, and (4) is again solved with respect to $k_1(\mathbf{x}_2)$. Since $k_2(\mathbf{x}_2) = 1/\nu_2$, the next correction factor is $\nu_3 = k_1(\mathbf{x}_2)\nu_2$. If convergence to a fixed correction factor value ν is obtained by iterative application of this calculation scheme, then the limit point $\mathbf{x}$ is a solution to both (1) and (9). However it is not guranteed that $\mathbf{x}$ is a point of tangency between the surface defined by (9) (on which it is a beta point, by definition) and the surface defined by (1). Several examples have shown that the deviation from tangency is usually small if the simple limit state surface has some similarity with the elaborate limit state surface in the critical region around the beta point, and even before the first iteration quite good estimates of the geometric reliability index can be obtained. In any case, the obtained geometric reliability index β_2 of the finally corrected simple limit state 2 is an upper bound on the geometric reliability index β_1 of the limit state 1. A further correction to this approximation can be made by computing the angle between the unit normals to the idealized and realistic limit-state surfaces, as described in (Franchin et al. 2000).

3 APPLICATION TO PROBLEMS INVOLVING INTEGRALS OF NON GAUSSIAN RANDOM FIELDS

Consider a real-valued scalar random field

$$Z(\mathbf{x}), \quad \mathbf{x} \in \Omega \subseteq \Re^n \tag{10}$$

defined on the subdomain Ω of the n-dimensional real space, with mean function $\mu_Z(\mathbf{x})$, standard deviation function $\sigma_Z(\mathbf{x})$ and auto-correlation coefficient function $\rho_{ZZ}(\mathbf{x}_1, \mathbf{x}_2)$. Of special consideration in this paper are Nataf-type fields, which are defined through a translation of a standard Gaussian field (of zero mean and unit variance) so as to have a prescribed marginal cumulative distribution function $F_Z(z, \mathbf{x})$. The transformation has the form $Z(\mathbf{x}) = F^{-1}[\Phi(U(\mathbf{x})), \mathbf{x}]$, where $F^{-1}(\cdot, \mathbf{x})$ is the inverse of $F_Z(z, \mathbf{x})$ with respect to the first argument, $\Phi(\cdot)$ is the standard normal cumulative distribution function, and $U(\mathbf{x})$ is the Gaussian field. It can be shown (Liu & Der Kiureghian) that the auto-correlation coefficient functions of the two fields are related through the integral equation

$$\rho_{ZZ}(\mathbf{x}_1, \mathbf{x}_2) = \iint \frac{z(\mathbf{x}_1) - \mu(\mathbf{x}_1)}{\sigma(\mathbf{x}_1)} \, \frac{z(\mathbf{x}_2) - \mu(\mathbf{x}_2)}{\sigma(\mathbf{x}_2)} \, \phi_2[u_1, u_2, \rho_{UU}(\mathbf{x}_1, \mathbf{x}_2)] \, du_1 \, du_2 \tag{11}$$

where $\phi_2(\cdot, \cdot, \rho)$ is the bi-variate standard normal density having the correlation coefficient ρ and $z(\mathbf{x}_i) = F^{-1}[\Phi(u_i), \mathbf{x}_i]$. In general the difference between $\rho_{ZZ}(\mathbf{x}_1, \mathbf{x}_2)$ and $\rho_{UU}(\mathbf{x}_1, \mathbf{x}_2)$ is not large. However, whereas the positive-definiteness of $\rho_{UU}(\mathbf{x}_1, \mathbf{x}_2)$ guarantees the same property for $\rho_{ZZ}(\mathbf{x}_1, \mathbf{x}_2)$, the reverse is not necessarily true. (Lack of positive definiteness of $\rho_{UU}(\mathbf{x}_1, \mathbf{x}_2)$ implies that the Nataf model is not valid for the prescribed marginal distribution and auto-correlation coefficient function.) Hence, in application it is convenient to specify $\rho_{UU}(\mathbf{x}_1, \mathbf{x}_2)$ and then compute $\rho_{ZZ}(\mathbf{x}_1, \mathbf{x}_2)$ from the above relation. Approximate formulas relating the two correlation coefficients are given in (Liu & Der Kiureghian 1986, Ditlevsen & Madsen 1996).

Now consider the integral

$$\mathcal{I} = \int_D \psi(\mathbf{x}) \, Z(\mathbf{x}) \, d\mathbf{x}, \quad D \subseteq \Omega \tag{12}$$

over the domain D, where $\psi(\mathbf{x})$ is a deterministic function (analogous to a shape function in finite element analysis).

If the field is Gaussian, then $\mathcal{I}$ is also Gaussian. However, when the field is non-Gaussian, the probability distribution of $\mathcal{I}$ is unknown, even when the full probabilistic description of the field is given. In this case, if the integration domain D is sufficiently large with respect to a characteristic correlation scale of the field, and the mean and variance of the modulated field $\psi(\mathbf{x})Z(\mathbf{x})$ vary slowly with respect to the correlation scale, then some generalized form of the central limit theorem may justify the assumption that the distribution of $\mathcal{I}$ is approximately Gaussian. In fact, the integral is then the sum of a large number of approximately independent and identically distributed random variables. When the field is homogeneous and $\psi(\mathbf{x})$ is a constant the random variables are truly identically distributed and the only relevant factor is the size of the integration domain relative to the correlation scale of the field. On the other hand, if the size of D is sufficiently small, the distribution shape of $\mathcal{I}$ is, under suitable continuity assumptions about the field and the function $\psi(\mathbf{x})$, approximately of the same shape as the shape of the marginal distribution of the field. The distribution parameters of course change in accordance with integration of the mean value function and the covariance function of the field.

Subdividing the integration domain D into a union of N mutually exclusive and collectively exhaustive sub-domains d_i, the integral can be written as

$$\mathcal{I} = \sum_{i=1}^{N} \int_{d_i} \psi(\mathbf{x}) \, Z(\mathbf{x}) d\mathbf{x} = \sum_{i=1}^{N} \mathcal{I}_i \tag{13}$$

The first and second moments of $\mathcal{I}_i$, $i = 1, \ldots, N$, are readily obtained by integration over the appropriate domains of the mean and covariance functions

of the field. If the sub-domains d_i are sufficiently small, as mentioned above, the marginal distribution of each sub-integral $\mathcal{I}_i$ is approximately of the same type as the marginal distribution of the field within the sub-domain. The joint distribution of the set of sub-integrals is specified by the Nataf distribution, as described earlier.

Now consider a reliability problem having a realistic limit-state function

$$g_{\mathrm{r}}(\mathcal{S}_1, \ldots, \mathcal{S}_m, \mathcal{R}_1, \ldots, \mathcal{R}_n, \mathbf{D}) \tag{14}$$

where $\mathcal{S}_i$, $i = 1, \ldots, m$, and $\mathcal{R}_j$, $j = 1, \ldots, n$, are integrals of load and resistance random fields, respectively, of the type described above and $\mathbf{D}$ is a vector that collects all the remaining random variables. One way to solve this reliability problem is to replace each of the integrals by a sum of sub-integrals as in (13) such that the limit-state function reads

$$g_{\mathrm{r}}\Big(\sum_{i=1}^{N_{\mathcal{S}_1}} \mathcal{S}_{1i}, \ldots, \sum_{i=1}^{N_{\mathcal{S}_m}} \mathcal{S}_{mi}, \sum_{i=1}^{N_{\mathcal{R}_1}} \mathcal{R}_{1i}, \ldots, \sum_{i=1}^{N_{\mathcal{R}_n}} \mathcal{R}_{ni}, \mathbf{D}\Big) \tag{15}$$

The random variables of the problem then are the set of all the sub-integrals $\mathcal{S}_{1i}$, $\mathcal{R}_{1i}$, etc., and $\mathbf{D}$, which can be a considerably large set. The conventional FORM analysis requires repeated computations of the limit-state function and its gradient with respect to the random variables. With the large number of random variables involved, such an analysis could easily become prohibitive - hence the motivation for the MCFM approach to solve this problem.

For the MCFM solution of the problem, we select the idealized limit-state function of the same form as in (14), but assign the normal distribution to the integrals $\mathcal{S}_i$ and $\mathcal{R}_i$. That is, we select the limit state function

$$g_{\mathrm{i}}(\mathcal{S}_{\mathrm{i}1}, \ldots, \mathcal{S}_{\mathrm{i}m}, \mathcal{R}_{\mathrm{i}1}, \ldots, \mathcal{R}_{\mathrm{i}n}, \mathbf{D}) \tag{16}$$

where the subscript i signifies the idealized nature of the limit-state function and the individual integrals. The MCFM solution of the problem proceeds as follows: We first solve the reliability problem defined by (16) (analogous to the solution of (9) with $\nu^* = \nu_1 = 1$). Note that the random variables representing the load and resistance integrals here are Gaussian and a conventional FORM approach can be used to easily find the beta point. Let $s^*_{\mathrm{i}i}$ and $r^*_{\mathrm{i}i}$ be the coordinates of the beta point for the load and resistance integrals. We now need to split these integral values in order to have the same dimension of variables as in the realistic model (15). This is done by finding beta points for the $m + n$ linear limit-state equations

$$\mathcal{S}_{\mathrm{i}i1} + \ldots + \mathcal{S}_{\mathrm{i}iN_{\mathcal{S}_i}} - s^*_{\mathrm{i}i} = 0, \;\; i = 1, \ldots m \tag{17}$$

$$\mathcal{R}_{\mathrm{i}i1} + \ldots + \mathcal{R}_{\mathrm{i}iN_{\mathcal{R}_i}} - r^*_{\mathrm{i}i} = 0, \;\; i = 1, \ldots n \tag{18}$$

where $\mathcal{S}_{\mathrm{i}ij}$ and $\mathcal{R}_{\mathrm{i}ij}$ are the idealized counterparts of $\mathcal{S}_{ij}$ and $\mathcal{R}_{ij}$. Since these idealized random variables are Gaussian and the equations (17) and (18) are linear in the random variables, the beta points are easily obtained by use of the standard linear regression formula. This yields coordinates $s^*_{\mathrm{i}ij}$ and $r^*_{\mathrm{i}ij}$ of the beta point in the augmented space of the sub-integral random variables. These coordinates are now transformed to the standard normal space and from there back to the space of the original non-Gaussian sub-integrals. Let s^*_{ij} and r^*_{ij} denote these coordinates. (Note that we have removed the subscript i since these are now the realistic variables.) Next we solve the equation

$$g_{\mathrm{r}}\Big(k_1 \sum_{i=1}^{N_{\mathcal{S}_1}} s^*_{1i}, \ldots, k_1 \sum_{i=1}^{N_{\mathcal{S}_m}} s^*_{mi}, \sum_{i=1}^{N_{\mathcal{R}_1}} r^*_{1i}, \ldots, \sum_{i=1}^{N_{\mathcal{R}_n}} r^*_{ni}, \mathbf{D}\Big) = 0 \tag{19}$$

for k_1. This gives the new value $\nu_2 = k_1$ for the model correction factor. The next cycle of the iteration begins by solving the idealized problem in (16) while replacing $\mathcal{R}_{\mathrm{i}i}$ by $\nu_2 \mathcal{R}_{\mathrm{i}i}$. After finding the beta point, performing the split and transforming into the non-Gaussian space, a new value of k_1 is computed from (19). The next trial value of the model correction factor is $\nu_3 = k_1 \nu_2$. This process is continued until some chosen stop criterion is satisfied. In case the sequence $\nu_1, \nu_2, \ldots$ converges to the limit $\nu*$, the corresponding beta point for the idealized problem represents the MCFM approximation for the beta point of the realistic problem in the sense described in Section 2.

It is important to realize that the above approach only requires repeated computations of the realistic limit-state to solve for k_1 in (19). No gradient computations with the realistic limit-state function (15) are necessary.

4 APPLICATION: STABILITY OF PILE FOUNDATION IN CLAYEY SOIL

We consider the stability of a pile foundation in a layered, friction-less, cohesive soil under a load from its superstructure. These conditions may arise in a clayey soil in undrained conditions that is subjected to a rapidly applied load, with the time scale measured relative to the rate of consolidation in the soil that is related to its permeability. After formulation of the model, we investigate the reliability of a single pile and then the system reliability of a pile group.

4.1 *Load bearing capacity of a single pile*

A foundation pile transfers the load to the surrounding soil through shear stresses on its lateral surface

(commonly called skin or shaft resistance) and compressive stresses at its tip. For the sake of simplicity of the presentation, we neglect the contribution from the tip and only consider the capacity due to the shear stresses on the lateral surface of the pile. Neglecting the weight of the pile, the ultimate capacity of the pile can be written as

$$R_u = \sum_{i=1}^{n} \int_{D_i} \pi d C_{u,i}(x) dx \tag{20}$$

where the summation is over the soil layers, d denotes the diameter of the pile, x denotes the depth coordinate, and $C_{u,i}(x)$ denotes the undrained cohesion field of the i^{th} soil layer. Since cohesion is non-negative, each $C_{u,i}(x)$ must be modeled as a non-Gaussian field. Therefore, the seemingly simple problem of the reliability of a single pile defined by the linear limit-state function

$$g(R_u, P) = R_u - P \tag{21}$$

where P is the applied load, cannot be solved by conventional reliability methods since it involves integrals of non-Gaussian fields.

4.2 *Models of the cohesion random field*

From the inspection of CPT profiles, it is evident that the undrained cohesion of most soils is a non-homogeneous random field. Typically, both the mean and the standard deviation increase with depth, but the coefficient of variation remains more or less constant. For the analysis here, we formulate two such fields with lognormal and beta marginal distributions. In the following, the subscript u, i is dropped and the random cohesion field for a given soil layer is denoted as $C(x)$.

A non-homogeneous lognormal cohesion field with a constant c.o.v. is defined by

$$C(x) = e^{\lambda(x) + \zeta U(x)} \tag{22}$$

where $U(x)$ is a homogeneous Gaussian field of zero mean and unit standard deviation. The function $\lambda(x)$ and ζ are defined in terms of the target mean function $\mu_C(x)$ and constant c.o.v. δ_C of the the lognormal field. Specifically, $\zeta = \sqrt{\log(1+\delta_C^2)}$ and $\lambda(x) = \log[\mu_C(x)] - \zeta^2/2$. Furthermore, the correlation functions of the two fields are related through (11), which in the present case has the closed form solution

$$\rho_{CC}(x_1, x_2) = \frac{1}{\delta_C^2}[e^{\zeta^2 \rho_{UU}(x_1, x_2)} - 1] \tag{23}$$

As mentioned earlier, to guarantee the positive definiteness of the correlation function of the lognormal field, the correlation structure is assigned to the standard Gaussian field $U(x)$ and the corresponding one for $C(x)$ is computed according to (23). The particular correlation function

$$\rho_{UU}(\mathbf{x}_1, \mathbf{x}_2) = e^{-\frac{|\mathbf{x}_1 - \mathbf{x}_2|}{c}} \tag{24}$$

is used in the numerical examples.

A non-homogeneous Beta cohesion field with a constant c.o.v. is defined by

$$C(x) = \mu_C(x)[1 + V(x)] \tag{25}$$

where $V(x)$ is a zero-mean homogeneous Nataf field having a symmetric beta marginal distribution defined on $[-1, 1]$. $V(x)$ is non-dimensional and represents the variation of the cohesion. It is a simple matter to show that the above field has the beta distribution with the mean function $\mu_C(x)$, standard deviation function $\mu_C(x)\sigma$ and a constant c.o.v. $\delta_C = \sigma$, where σ denotes the standard deviation of $V(x)$. Furthermore, we have $\rho_{CC}(x_1, x_2) = \rho_{VV}(x_1, x_2)$, where $\rho_{VV}(x_1, x_2)$ is obtained from (11) for the assumed correlation function of the underlying Gaussian field.

4.3 *Extension to pile group*

When a group of piles is considered various problems arise concerning the overall behavior of the foundation. For the purpose of showing the proposed method at work, we will use a simplified model that does not take into account such factors as the effectivity of the pile group (usually measured in terms of the ratio between the actual load bearing capacity of the pile group to the sum of the individual pile capacities), the contribution from the foundation mat that is in contact with the ground, or the non-linear force-deformation behavior of the pile-soil system. The simplified model assumes rigid-plastic behavior of the soil and hinged connections of the piles to the foundation mat. Considering the virtual displacements for the planar pile system in Figure1, the difference between the internal and external virtual works

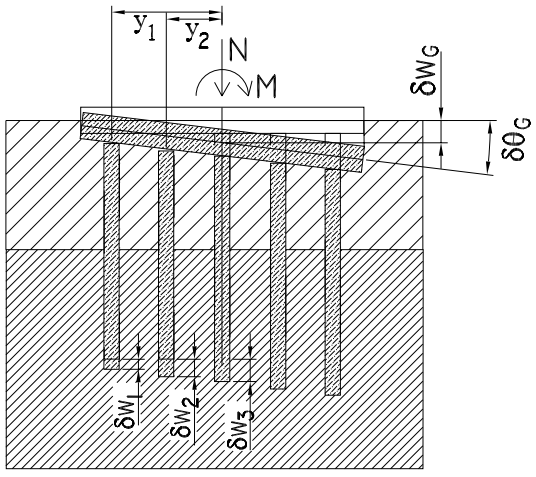

Figure 1. Pile group: virtual displacement field.

of the system is given by

$$\delta W_{\text{int}} - \delta W_{\text{ext}} = \sum_{i=1}^{n_p} R_{ui}\delta w_i - N\delta w_G - M\delta\theta_G \tag{26}$$

where R_{ui} is the capacity of the i^{th} pile and N and M are the applied normal force and bending moment on the foundation mat, respectively. From the geometry in Figure1, we have $\delta w_i = \delta w_G + \delta\theta_G y_i$. Substituting this in (26) and setting the coefficients of δw_G and $\delta\theta_G$ equal to zero, we obtain the two limit-state functions

$$g_{\text{trans}}(R_1, \ldots, R_{n_p}, N) = \sum_{i=1}^{n_p} R_{ui} - N \tag{27}$$

$$g_{\text{rot}}(R_1, \ldots, R_{n_p}, M) = \sum_{i=1}^{n_p} R_{ui}y_i - M \tag{28}$$

which represent the translation and rotation failure modes, respectively. Thus, the problem is that of a series system reliability with two failure modes.

In this particular case of two failure modes, the FORM approximation to the system failure probability can be determined explicitly when the FORM solutions for the two component events are known. Given the reliability indices $\beta_1 = \beta_{\text{trans}}$, $\beta_2 = \beta_{\text{rot}}$ and the corresponding normal vectors $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, the system failure probability is

$$p_{\text{system}} = 1 - \Phi_2(\beta_1, \beta_2, \rho_{12}) \tag{29}$$

where $\Phi_2(\beta_1, \beta_2, \rho_{12})$ is the bivariate normal distribution function corresponding to zero means, unit variances and correlation coefficient $\rho_{12} = \boldsymbol{\alpha}_1^{\mathsf{T}}\boldsymbol{\alpha}_2$.

4.4 *Results*

First consider a single pile of diameter $d = 1$, length $D = 10$, one soil layer with a random cohesion field having the mean function $\mu_C(x) = 10 + x$ and the constant c.o.v. $\delta_C = 0.5$, and a random load P having the Gaussian distribution with mean $\mu_P = 200$ and standard deviation $\sigma_P = 50$.

Parametric studies are carried out for both field types in order to investigate the behavior of the approximate solution given by the MCFM with respect to the correlation length of the field (measured in terms of parameter c in (24)) and the number N of subdivisions of the integral. The results for the reliability index β and the correction factor ν^* are shown in Figures 2 to 9. It is seen that β decreases with increasing correlation length (Figures 2 and 4), and that ν^* approaches 1 as the correlation length approaches zero (Figures 3 and 5). The first is due to the fact that the variance of the resistance integral increases

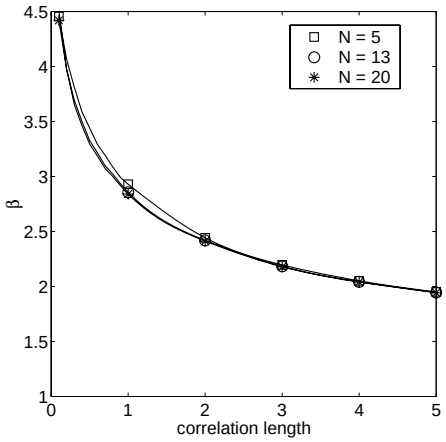

Figure 2. Lognormal field: Variation of β with correlation length parameter c and number of integral subdivisions N.

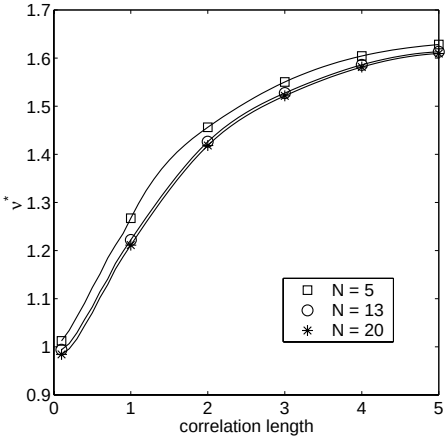

Figure 3. Lognormal field: Variation of ν^* with correlation length parameter c and number of integral subdivisions N.

with increasing correlation length, while its mean remains constant. The second finding is a consequence of the central limit theorem: for a short correlation length, the sub-integrals are nearly statistically independent and their sum is nearly Gaussian, hence the idealized model is nearly accurate and we find that ν^* approaches 1.

Figures 6 and 7 compare the results for the two distributions for the case with $N = 20$. Also shown in Figure 6 is the reliability index when the cohesion field is assumed to be Gaussian, even though such an assumption is not consistent with the fact that cohesion is nonnegative. As the correlation length approaches zero, all three distributions yield the same reliability index $\beta = 4.41$ (Figure 6) and both correction factors converge to $\nu^* = 1$ (Figure 7). This shows that, for a short correlation length, the resistance integral is practically Gaussian and that the reliability index is independent of the distribution of the field. For larger values of the correlation length, the β values for the three distribution types are significantly different, indicating the strong dependence of the reliability index on the field distribution. Interestingly, for such correlation lengths, the ν^* values for the two distribution types depart from 1 in opposite directions: $1 < \nu^*$ for the lognormal distribution and $\nu^* < 1$ for the beta distribution.

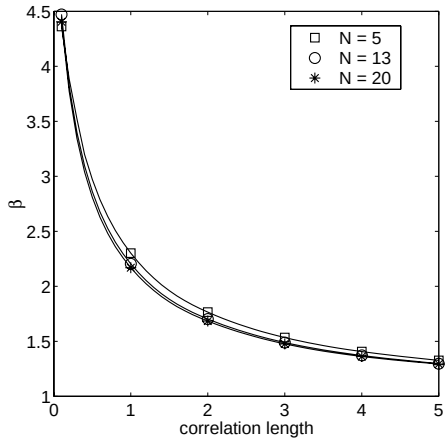

Figure 4. Beta field: Variation of β with correlation length parameter c and number of integral subdivisions N.

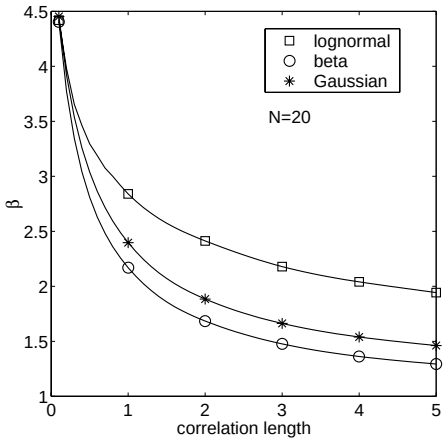

Figure 6. Influence of the field distribution on the reliability index.

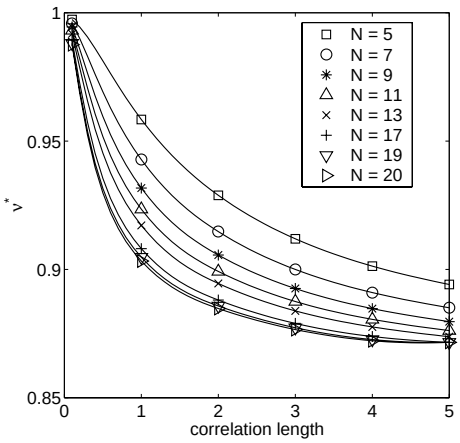

Figure 5. Beta field: Variation of ν^* with correlation length parameter c and number of integral subdivisions N.

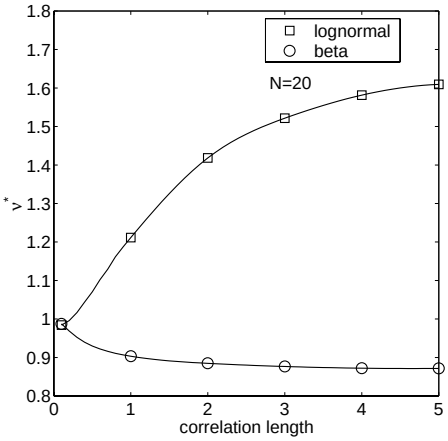

Figure 7. Correction factor ν^* for two field distributions.

Finally, comparisons with "exact" FORM results are presented in Figures 8 and 9 for the two distributions with $N = 20$. Shown are exact beta values computed by directly solving the realistic limit-state function, the (zeroth order) MCFM results, and results obtained by correcting for the angle ω between the normals to the realistic and the idealized limit state surfaces at the beta point of the idealized surface in the standard Gaussian space. Two methods are used to compute ω: uniform sampling on a cone, as described in (Ditlevsen 1996), and direct gradient computations. Of course, the second method is impractical when the number of sub-integrals is large.

Figure 8 for the lognormal field shows that the (zeroth order) MCFM method produces fairly good results for all correlation lengths. The corrected result based on the sampling estimation of ω is almost coincident with the exact result and the result based on the computation of ω using the gradients. Figure 9 for the Beta field shows excellent agreement between all four results.

Now consider a foundation mat supported by three piles in a plane and loaded by a Gaussian overturning moment M of mean $\mu_M = 1000$ and standard deviation $\sigma_M = 200$, and a Gaussian vertical force N of mean $\mu_N = 900$ and standard deviation $\sigma_N = 200$. The horizontal coordinates of the piles relative to the point of application of the vertical load are $y_1 = -5$, $y_2 = 0$ and $y_3 = 5$.

All piles have the common diameter $d = 1$ and length $D = 10$. A two-dimensional lognormal cohesion field is assumed where the corresponding standard Gaussian field has the orthotropic correlation function $\rho_{UU}(\mathbf{x}_1, \mathbf{x}_2) = \exp(-|x_1 - x_2|/c_x - |y_1 - y_2|/c_y)$, in which c_y is the correlation length in the horizontal direction and c_x is the correlation length in the vertical direction. Owing to the slower variation of soil properties in the horizontal direction, $c_y = 5$ is assumed, whereas c_x is varied from 0.1 to 5. Figure 10 summarizes the results of system reliability analysis of the pile group by the proposed MCFM method. Shown are the reliability indices for each failure mode and the generalized reliability index for the system. Both failure modes are found to contribute to the system failure probability.

5 CONCLUSIONS

An efficient method for approximately solving reliability problems involving integrals of non-Gaussian random fields is developed. The method makes use of the Model Correction Factor Method to obtain an approximation to the first-order reliability index by performing calculations on an idealized limit-state func-

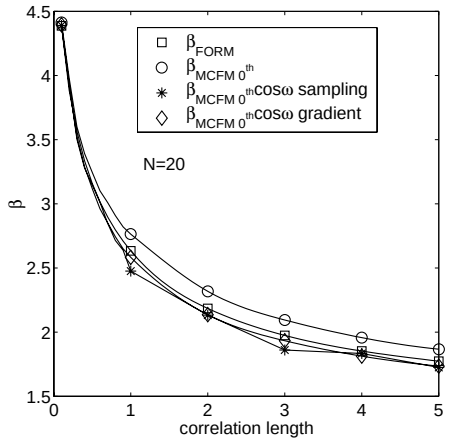

Figure 8. Lognormal field: Comparison with exact FORM results.

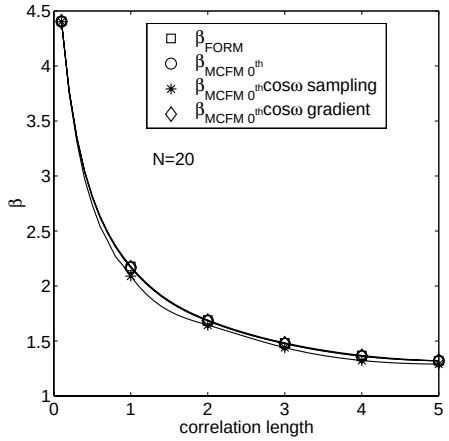

Figure 9. Beta field: Comparison with exact FORM results.

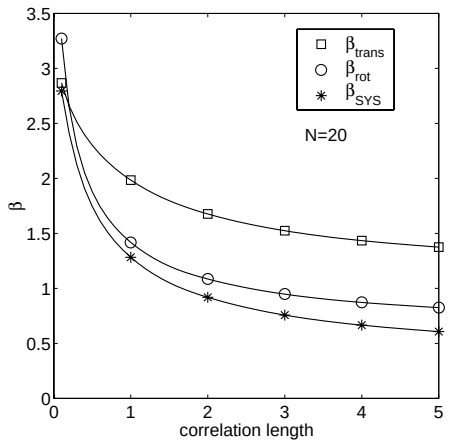

Figure 10. Component and system reliability indices for the pile group with lognormal cohesion field.

tion involving Gaussian random fields. A further correction is achieved by an efficient sampling method on a cone. Although in principle applicable to general non-Gaussian fields, the method is particularly effective in connection with Nataf fields, which are defined through monotonic marginal translations of Gaussian fields.

Investigations with simulated one-dimensional Nataf fields are carried out to determine the appropriate length of sub-divisions of the integration domain for which the distribution of the sub-integral remains the same as that of the field with sufficient accuracy, and for which the resulting correlation matrix of the vector of sub-integrals is non-singular. Through these studies, reported in (Franchin et al. 2000) but omitted herein due to limitation of number of pages, it is determined that the first condition is satisfied when the sub-interval length is no more than one-half to 5 times the correlation length of the field, depending on the type of distribution, and that the second condition is satisfied when the sub-interval length is no less than a third of the correlation length of the field. Thus, both conditions can be satisfied for any distribution type and correlation length by proper selection of the size of the integration sub-domains.

The proposed method is illustrated through its application to a pile foundation problem, in which the cohesion field of the soil layer is modeled as a non-Gaussian field. Lognormal and beta fields are considered. The analysis serves to demonstrate the accuracy of the proposed method as well as the significance of the non-Gaussian nature of the cohesion fields on the reliability of the pile foundation.

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