# Gradient angle estimation by uniform directional simulation on a cone *

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**Abstract.** A sample of uniformly distributed unit vectors on an $n$-dimensional spherical cone is generated. The distances to a given limit state surface in the directions of the unit vectors of the sample are calculated and each of these distances are projected on the cone axis. The theoretical distribution of these projections is derived assuming the limit-state surface to be a hyperplane. This distribution depends on the angle between the cone axis and the normal vector to the hyperplane. Assuming sufficient flatness of the actual limit-state surface within a neighbourhood of the cut point with the cone axis, the cone top angle can be chosen small enough that this distribution can be taken as the basis for the formulation of the likelihood function of the angle given the sample of projections. The angle of maximum likelihood is then the indicator of whether the cut point can be taken as a sufficiently accurate approximation to a locally most central limit state point. Moreover, the estimated angle can be used to correct the geometric reliability index.

**Keywords:** Directional simulation, effectivity factor, gradient angle estimation, maximum likelihood, model-correction-factor method, Monte Carlo simulation, most central Limit-state point, beta-point check, reliability index correction.

## 1 Introduction

Calculations of gradients most often consume the essential part of the calculation time used for first order reliability analysis with respect to elaborate limit-state surfaces in spaces of high dimension. Iterated gradient calculations needed in the search for locally or globally most central limit state points in the standard Gaussian space can be prohibitive for terminating the search within practicable time limits. Therefore explicit so-called response surface approximations are often used as substitutes for elaborate limit-state surfaces.

A useful response surface is of a sufficiently simple mathematical form to allow a fast determination of the most central point on the response surface itself. The mathematical expression of the response surface contains a finite number of

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parameters that are given values such that the approximation error defined in some convenient way becomes minimal over a finite number of located points on the limit-state surface. Having obtained the most central point $B$ on the response surface, the limit state point $C$ on the line from the origin through $B$ is a reasonable candidate to take as an approximation to a locally most central limit state point $D$. Assuming that the limit-state surface is differentiable in a neighbourhood of $C$, the check of whether $C$ could be a stationarity point of the distance from the origin to the limit-state surface is to determine the direction of the normal vector to the limit-state surface at $C$. If the direction is coincident with the direction from the origin to $C$, then $C$ is certainly a stationarity point and possibly a locally most central limit state point $D$. If the direction only deviates by a small angle from the direction from the origin, a simple approximation to the geometric reliability index (Hasofer-Lind) can be obtained as the product of the distance to $C$ from the origin and the cosine to the angle.

This paper shows that a particular directional simulation method can be used to estimate the deviation angle reasonably well on the basis of maximum likelihood estimation applied to a sample of a suitably defined sampling variable. Even in spaces of high dimension this simulation method gives rather accurate angle estimates only using a very small sample size as compared to the dimension of the space.

Such a property is what is needed for a practicable check procedure in connection with response surface methods where calculation of limit state points are elaborate. In particular, the so-called model-correction-factor method gives an illustrative example of the usefulness of the presented combined simulation and maximum likelihood estimation method for verifying that a considered point on the limit-state surface is a locally most central limit state point.

# 2 Random unit vector distributed uniformly on spherical cone

Consider a spherical cone with vertex at the origin of the $n$-dimensional real space and with axis in the direction of the unit vector $\boldsymbol{e}_1$ of the first coordinate axis. Let the angle between $\boldsymbol{e}_1$ and any generatrix of the cone be $\gamma$. Moreover, let $\mathbf{X} = (X_1, \ldots, X_n)$ be an $n$-dimensional standard Gaussian random vector. Then the random directional unit vector

$$\boldsymbol{A} = \frac{\mathbf{Z}}{\|\mathbf{Z}\|} \tag{1}$$

where

$$\mathbf{Z} = \boldsymbol{e}_1 + \frac{\mathbf{X} - X_1\boldsymbol{e}_1}{\|\mathbf{X} - X_1\boldsymbol{e}_1\|} \tan\gamma \tag{2}$$

has a uniform distribution on the intersection between the cone and the unit sphere. Thus it is quite simple to simulate realizations of $\boldsymbol{A}$. By simple rotation

of the coordinate system the axis of the cone can be directed in any specified direction of the space. Therefore it is sufficient in the following to let the axis direction be coincident with the unit vector $\boldsymbol{e}_1$. We will denote this simulation as uniform unit vector sampling on the $\gamma$-cone in direction $\boldsymbol{e}_1$.

# 3 Directional distance sampling on a cone

Let $\mathfrak{F} \subset \mathbb{R}^n$ be a set with a boundary $\partial\mathfrak{F}$ that is cut at most at a single point by any generatrix of the $\gamma$-cone in direction $\boldsymbol{e}_1$. The distance to the cut point from the origin in the random direction $\boldsymbol{A}$ written as $r(\boldsymbol{A}\,)$ is obtained by solving the equation $g[r(\boldsymbol{A}\,)\boldsymbol{A}\,] = 0$, where $g(\mathbf{x}) = 0$ is the equation of the boundary surface $\partial\mathfrak{F}$. If, in particular, the surface $\partial\mathfrak{F}$ is a hyperplane orthogonal to $\boldsymbol{e}_1$ at the distance $d$ from the origin, then all values of the sampling variable

$$S = \frac{1}{d} r(\boldsymbol{A}\,) \cos\gamma \tag{3}$$

are 1. However, if $\partial\mathfrak{F}$ is a hyperplane through the point $d\boldsymbol{e}_1$ without being orthogonal to $\boldsymbol{e}_1$ then $S$ will show random deviation from 1. In fact, the distribution of $S$ gives information about the normal vector to $\partial\mathfrak{F}$ at the point $d\boldsymbol{e}_1$ given that this point is a point of $\partial\mathfrak{F}$ and that $\partial\mathfrak{F}$ is flat at this point (in the sense that FORM and SORM give approximately the same generalized reliability index if the normal vector to $\partial\mathfrak{F}$ at the point should happen to be coincident with $\boldsymbol{e}_1$). Assume that this flatness condition allows to replace the surface $\partial\mathfrak{F}$ with a hyperplane that has the normal unit vector $\mathbf{n}$ at an angle $\omega$ with $\boldsymbol{e}_1$. Moreover, let $\gamma$ be chosen such that $\gamma \ll \frac{\pi}{2} - \omega$. As noted, $S$ is the constant 1 if and only if $\omega = 0$.

The density function of $r(\boldsymbol{A}\,)$ is determined by using the rotational symmetry of the density af the directional vector $\boldsymbol{A}$ with respect to $\boldsymbol{e_1}$. It follows that $r(\boldsymbol{A}\,)$ irrespective of the direction of the unit normal vector $\mathbf{n}$, given the angle $\omega$, has the same probability distribution as if $\mathbf{n}$ is fixed to be the vector $\mathbf{n} = \cos\omega\ \boldsymbol{e_1} + \sin\omega\ \boldsymbol{e_2}$.Thus

$$r(\boldsymbol{A}\,) =_d \frac{d\cos\omega}{A_1\cos\omega + A_2\sin\omega} \tag{4}$$

where $A_1 = \cos\gamma$ and $=_d$ means equal in distribution. The second coordinate $A_2$ of $\boldsymbol{A}$ obviously has a density on the interval $]-\sin\gamma, \sin\gamma[$ obtained by projecting the uniform density on the surface of the $(n-1)$-dimensional sphere of radius $\sin\gamma$ on a diameter line of the sphere. By a geometric argument it is thus seen that the density of $A_2$ is

$$f_{A_2}(x) = \frac{\left[1 - \left(\frac{x}{\sin\gamma}\right)^2\right]^{\frac{n-4}{2}}}{2\sin\gamma \int_0^1 (1-t^2)^{\frac{n-4}{2}} dt}, \quad -\sin\gamma < x < \sin\gamma \tag{5}$$

The density of $r(\boldsymbol{A}\,)$ is hereafter directly obtainable from the relation (4) as

$$\begin{aligned} f_{r(\boldsymbol{A}\,)}(r) &= f_{A_2}(a_2) \mid \frac{da_2}{dr} \mid = f_{A_2}\big[\big(\frac{d}{r} - cos\gamma\big)\frac{1}{\tan\omega}\big]\frac{d}{r^2\tan\omega} \\ &= \frac{\big(\frac{d}{r}\big)^2\big[1-\big(\frac{\frac{d}{r}-cos\gamma}{\sin\gamma\tan\omega}\big)^2\big]^{\frac{n-4}{2}}}{2d\tan\omega\sin\gamma\int_0^1(1-t^2)^{\frac{n-4}{2}}dt},\ \frac{\cos\omega}{\cos(\omega-\gamma)} \le \frac{r}{d} \le \frac{\cos\omega}{\cos(\omega+\gamma)} \end{aligned} \tag{6}$$

where-upon the density of the sampling variable $S$ is obtained from (3) as

$$f_S(s) = \frac{\big(\frac{1}{s}\big)^2\big[1-\big(\frac{\frac{1}{s}-1}{\tan\gamma\tan\omega}\big)^2\big]^{\frac{n-4}{2}}}{2\tan\gamma\tan\omega\int_0^1(1-t^2)^{\frac{n-4}{2}}dt},\ 1-\tan\gamma\tan\omega \le \frac{1}{s} \le 1+\tan\gamma\tan\omega \tag{7}$$

# 4 Maximum likelihood estimation of the angle

Let $s_1,\dots,s_m$ be a sample of $S$ using a sample $\mathbf{z_1},\dots,\mathbf{z_m}$ of the random vector $\mathbf{Z}$ generated by use of (2). The loglikelihood function for $\omega$ then becomes

$$L[\omega \mid s_1,\dots,s_m] = \sum_{i=1}^{m}\log[f_S(s_i \mid \omega)] \tag{8}$$

and it is defined for $0 \le \omega < \frac{\pi}{2} - \gamma$. A point estimate of $\omega$ is the point of maximal loglikelihood.

To test the applicability of this method of estimating the angular deviation of the normal vector $\mathbf{n}$ from the first axis unit vector $\boldsymbol{e}_1$, some example calculations are shown in the following. It is the idea of this simulation method, of course, that the number of calculations of $r(\boldsymbol{A})$ should be considerably less than that needed for deterministic numerical calculation of the normal vector, that is, considerably less than the dimension $n$ of the space. Within the range of small values of $\omega$ the method is anticipated to be effective simply because the sampling variable $S$ becomes the constant 1 for $\omega = 0$ and a plane surface $\partial\mathfrak{F}$.

This is the asymptotic verification situation often met when applying response surface methods for reliability analysis with elaborate limit-state surfaces in high dimensions. In the next section a particular response surface method is considered for demonstration of the need. This particular method has been called the model-correction-factor method because it is based on a simplified physical modelling of the limit state. By correction factoring it becomes a response surface of closer resemblance to the elaborate limit-state surface than an arbitrary non-physical second or third degree algebraic surface.

As the test example the hyperplane with the normal unit vector $\mathbf{n} = \cos\omega\boldsymbol{e}_1 + \sin\omega\boldsymbol{e}_2$ is taken. Then $r(\boldsymbol{A})$ is simply calculated by the right side of (4). Figure 1 shows the graphs of different loglikelihood functions corresponding to $\gamma = 0.1$, the space dimensions $n = 10, 50, 100, 500$, and angles $\omega$ corresponding to $\tan\omega = 0.05, 0.1, 0.2, 0.4, 0.8$. The sample sizes $m$ are given as $\frac{m}{n} = 0.02, 0.1, 0.2, 0.4, 0.8$

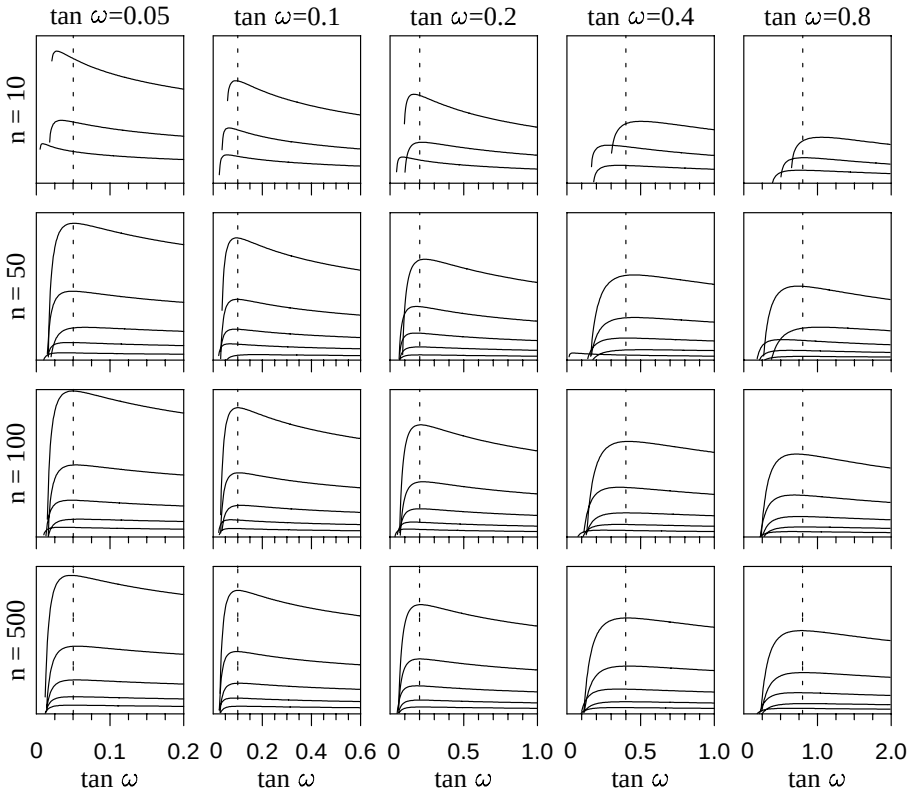


Figure 1: Graphs of loglikelihood functions for $\tan\omega$ obtained by test simulations with known angular deviation $\omega$ of hyperplane normal vector from first axis unit vector (marked by the vertical dotted lines). The space dimension is $n$ and the sample size $m$ is (top down) 0.8, 0.4, 0.2, 0.1, 0.02 times $n$ given $m > 1$ , respectively.

(given $m > 1$). The lowest graph in each diagram corresponds to the smallest sample size (required to be larger than 1, of course) and the graphs become placed in monotoneous order from bottom to top according to increasing sample size. Since only the determination of the positions of the maximum points on the $\tan\omega$-axes is of interest, no units are marked on the ordinate axes.

It is seen that this simulation method is quite effective as a tool to estimate the unknown angle $\omega$, in particular for the larger dimensions $n$ where the tool is actually needed. It is also seen that rather small sample sizes as compared to the dimension of the considered space are sufficient to obtain reasonably accurate estimates. The same test examples tried for $\gamma = 0.05$ and 0.2 show that the value of $\gamma$ has negligible influence on the maximum likelihood estimates of $\omega$.

# 5 Application to model-correction-factor method

The model-correction-factor method has been formulated and tested in a series of papers [1, 2, 3, 4]. The following short recapitulation of the method is closely following and partly citing the presentation given in [5].

For a given structure let $\mathbf{x}_F$ be the vector of all basic variables with physical units that contain the unit of force and let $\mathbf{x}_D$ be the vector of all remaining basic variables (of the type as geometric and dimensionless basic variables). With sufficient generality we can assume that the basic variables are defined such that the units of the elements of $\mathbf{x}_F$ are all proportional to the force unit. Let $\mathbf{x}_F = (\mathbf{x}_S, \mathbf{x}_R)$ de split into the vector $\mathbf{x}_S$ of load variables and the vector $\mathbf{x}_R$ of strength variables, respectively, and consider two limit-state equations between the vectors $\mathbf{x}_S$, $\mathbf{x}_R$, and $\mathbf{x}_D$ to be one defined by an elaborate model, and the other by a simple model formulated as an idealization of the elaborate model.

The input values are assumed to be specified as random vectors $(\mathbf{X}_S, \mathbf{X}_R, \mathbf{X}_D)$ with a given joint probability distribution. The quantity of interest is the probability that a realization of $(\mathbf{X}_S, \mathbf{X}_R, \mathbf{X}_D)$ is obtained in the failure set $\mathfrak{F}_r$ of the elaborate (r for "realistic") model. The problem at hand is that the calculation of this probability $P(\mathfrak{F}_r)$ is elaborate. Therefore it is attractive to try to take advantage of the simple model by which the probability $P(\mathfrak{F}_i)$ of getting a realization in the idealized failure set $\mathfrak{F}_i$ can be calculated with less effort than required for the calculation of $P(\mathfrak{F}_r)$.

The two limit states are defined as the set of zero points of the functions $g_r(\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D)$ and $g_i(\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D)$, respectively, where the first function is suitably regular but not necessarily given in explicit form, and the second function is a less elaborate function than the first. It is assumed that both the safe sets are star-shaped in terms of $\mathbf{x}_S$ with respect to the origin of $\mathbf{x}_S$, that is, for any $(\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D)$ each of the two equations

$$g_r(\kappa_r \mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D) = 0 \tag{9}$$

$$g_i(\kappa_i \mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D) = 0 \tag{10}$$

has a unique solution with respect to $\kappa_r$ and $\kappa_i$, respectively. Defining the so-called effectivity factor as

$$\nu(\mathbf{x}) = \frac{\kappa_r(\mathbf{x})}{\kappa_i(\mathbf{x})} \tag{11}$$

where $\mathbf{x} = (\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D)$, it is then a consequence of the physical property of dimension homogeneity of the limit-state equations that the limit states defined by each of the equations

$$g_r(\mathbf{x}_S, \mathbf{x}_R, \mathbf{x}_D) = 0 \tag{12}$$

$$g_i(\mathbf{x}_S, \nu(\mathbf{x})\mathbf{x}_R, \mathbf{x}_D) = 0 \tag{13}$$

are identical (see proof in [5]).

Being identical, the two equations are equally elaborate, of course. However, it is reasonable to expect that the effectivity factor locally can be approximated by a constant. Then (13) may as an approximation be replaced by the equation

$$g_i(\mathbf{x}_S, \nu^*\mathbf{x}_R, \mathbf{x}_D) = 0 \tag{14}$$

applicable in a more or less wide neighbourhood of any point $\mathbf{x}^*$ at which $\nu^* = \nu(\mathbf{x}^*)$ is calculated. Thus $\nu^*$ acts as a model-correction factor applied to the strength variables in the idealized model. For reliability analysis purposes the best choice of the value of this factor obviously is the one that is obtained at the most central point of the elaborate limit-state surface represented in the standard Gaussian space. Given that $\nu(\mathbf{x})$ actually has a point of stationarity at $\mathbf{x}^*$, that is, given that all the partial derivatives of $\nu(\mathbf{x})$ are zero at $\mathbf{x}^*$, then the two limit-state surfaces defined by (12) and (14) are tangential to each other at $\mathbf{x}^*$. Thus the two limit-state surfaces have $\mathbf{x}^*$ in common as a point that satisfies the necessary conditions for being a most central point also for (14).

The search for the point $\mathbf{x}^*$ with these properties may start by applying FORM to (14) with $\nu^*$ put to some judgementally chosen value, $\nu^* = 1$,say. This determines a first approximation to the most central point. At this point the equation (9) is solved with respect to $\kappa_r$, the solution being a new value of the correction factor. Using this factor in (14) a new FORM analysis gives a new approximation to the most central limit state point. At this new point, (9) is again solved with respect to $\kappa_r$, and the ratio between this solution and the previous solution is a new assessment of the correction factor. If convergence to a fixed correction factor value $\nu$ is obtained by iterative application of this calculation, then the limit point $\mathbf{x}$ is a solution to both (12) and (14). However, it is not sure that $\mathbf{x}$ is a point of tangency between the surface defined by (14) (on which it is a locally most central point, per definition) and the surface defined by (12). On the other hand, by an intelligent on physical principles based modelling of the idealized limit-state surface it is intended, of course, that it gets reasonable similarity with the elaborate limit-state surface within the domain of importance. Therefore it should be expected that the deviation from tangency is small in many cases. In fact, several examples have shown that even before the first iteration quite good estimates of the geometric reliability index (Hasofer-Lind) can be obtained.

A check and possible correction for being out of tangency can be made by replacing the effectivity factor $\nu(\mathbf{x})$ by its first-order Taylor expansion at the limit point $\mathbf{x}$. This requires a determination of the normal vector $\mathbf{n}$ to the elaborate limit-state surface at $\mathbf{x}$. A numerical determination takes as many calculations of points on the eleborate limit-state surface in the vicinity of $\mathbf{x}$ as the dimension of the space. This can be prohibitively time consuming.

To verify that there is tangency or only minor deviation from tangency it can therefore be advantageous as a practicable alternative to apply the simulation method presented in this paper. Given that the elaborate limit-state surface is almost flat at the most central limit state point and the angle between the two

limit-state surfaces is small,then an estimate of the geometric reliability index is

$$\beta = \|\mathbf{x}\| \cos \hat{\omega} \tag{15}$$

where $\hat{\omega}$ is the estimate of the angle $\omega$. By making simple test simulations as in Section 4 the smallest needed sample size that is likely to provide sufficient accuracy in a given application can be estimated before the start of time consuming calculations of points on the elaborate limit-state surface.

Relying on the similarity between the two surfaces, the flatness can be judged by comparing the reliability results obtained by applying both FORM (curvature independent results) and SORM (curvature dependent results) to the idealized limit-state surface. Another possibility is to test whether there are significant deviations of the simulated sample from the distribution (7) obtained from the hyperplane assumption.

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