SYSTEM EFFECTS INFLUENCING THE BENDING STRENGTH OF TIMBER BEAMS *

Ove Ditlevsen

Department of Structural Engineering and Materials

Technical University of Denmark. DK 2800 Lyngby, Denmark, od@bkm.dtu.dk

and

Bo Källsner

Swedish Institute for Wood Technology Research, Stockholm, Sweden, Bo.Kallsner@Tratek.se

Abstract. A stochastic model of hierarchical series system type for the bending strength of spruce beams is defined from the anticipation that the bending failure takes place at a cross-section with a defect cluster formed by knots or grain irregularities. The parameters of the model are estimated from measurements of the strengths of a large number of short test pieces cut from the beams such that judgementally each test piece contains only a single defect cluster. The test piece is spliced to stronger wood beam shafts in both ends. Due to the occurrence of a substantial number of splice failures in the total test series of 197 tests a special maximum likelihood estimation procedure is applied to estimate the parameters. Assuming that the estimated parameters are applicable in the series system model for the full uncut beams a theoretical bending strength distribution function is obtained in dependence of number of defect clusters within the span of constant bending moment loading. A strong test of the predictional power of the model is established by experiments with 54 long beams from the same population of beams from which the small test pieces were cut.

Keywords: Timber beam strength distributions. Defect defined strength of timber beams. Maximum likelihood estimation from defect data, Stochastic series system mechanism in timber strength. Hierarchical model for timber strength.

1 Introduction

Along its length a spruce timber beam can be divided into disjoint intervals of clear wood separated by shorter intervals that contain defects such as clusters of knots or grain deviations. The bending strengths of the beam pieces of clear wood are reasonably expected to be larger and of considerably less variability from piece to piece than the bending strengths of the smaller pieces with defects. In fact, it should be expected that the bending failure of a long beam that contains several defect intervals is triggered within the weakest of these defect intervals relative to the actual bending moment variation along the beam. Thus

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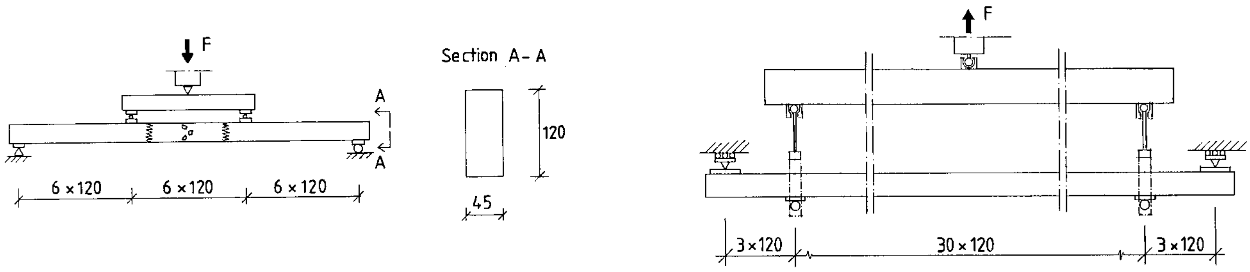

Figure 1: Left: Test arrangement for a single defect interval test piece which is finger jointed to stronger beam shafts. Right: Test arrangement for full uncut timber beams. All units are mm.

the bending strength mechanism should be expected to be a system mechanism that for a statically determinate beam is of a series system type.

To gain insight about the failure mechanism, two different series of carefully controlled experiments have been made at the Swedish Institute for Wood Technology Research in Stockholm. Details of these experiments are reported in [1, 2]. In short, a larger number of spruce trees have been cut in a specific area of a Swedish forest. Each of the logs have been cut into two twin beams under careful marking of their original position in the log and all beams have been dried together under controlled humidity and temperature conditions. One of the twin beams from each of 26 randomly chosen twin pairs was cut into shorter pieces such that each piece judgmentally only contained one defect interval. These short pieces were in both ends spliced together with beam pieces of considerably stronger wood each to form a beam that could be tested with respect to bending strength, Fig. 1 (left). Thereby a total data set of 197 strength values were obtained, grouped according to the positions of the beam pieces in the 26 original beams.

Before the experiments were made the model for statistical analysis of the data was planned to be a simple two-level hierarchical model. It was assumed that within the same beam there is a constant mean bending strength along the beam that varies randomly from beam to beam. Within the beam the defect intervals were assumed to have random zero mean bending strength deviations drawn independently from a common population for all beams. This hierarchical model is supported by indications in the literature about equicorrelation of bending strength for different cross-sections within the same beam. By suitable distribution assumptions and the maximum likelihood principle it should then be possible to estimate 26 individual mean bending strengths (not directly measured by the experiments, of course) and the variance of the population of bending strength deviations corresponding to the defect intervals.

However, it turned out that in 42 % of the experiments the failure occurred in one of the splice connections. Therefore the model for the statistical analysis of the data had to be modified to let the observed deviation from the mean strength within a beam be the smallest of three values. Two of these are the identically distributed splice strengths and the third is the strength of the defect interval. Thus the analysis turns to be a problem of inference from the observed strengths of 197 series systems each with three elements to say something about the strength of one specific element in the system. Including also the observation of the failure for each of the total number of 197 failures, the values of the model parameters are estimated by application of the maximum likelihood principle.

The results allow a prediction of the bending strength distribution of the uncut beams with their individual number of defect intervals. This is done by the supposition of putting the cut beam pieces together to form each of the original beams and by assuming that the failure is triggered within the weakest of the defect intervals of the beam. By bringing the corresponding twin beams to bending failure under constant bending moment along a major length of the beam, a data set is obtained against which the theoretical predictions can be tested, Fig. 1 (right). Also a series of bending strength experiments with 14 pairs of intact twin beams were made. The estimated significant correlation coefficient of about 0.68 between the strengths of the twins is shown to fit well with the considered hierarchical model from which the above mentioned equicorrelation coefficient is inferred also to be about 0.68, obtained solely on the basis of the experimental results for the spliced test beams.

2 Hierarchical series system model

From the description in Section 1 it follows that twin beams cut from a spruce log at each side of the pith typically have defects in the form of consecutive clusters of knots or local irregular grain patterns at about the same positions along the beams and at mutual distances that are of random size from beam pair to beam pair but of relatively small variability within the same pair of beams. Between these defects the beams consist of clear wood except for small separated knots of minor importance for the bending strength properties of the two beams. It should therefore be expected that the strength of the two beams are correlated while no correlation should be expected between the strengths of beams cut from logs from different trees given that the trees are all cut from a statistically homogeneous forrest area.

Having these features in mind, the following hierarchical model with two levels is one of the simplest realistic models to choose for describing the of bending strength in dependence of span and number of defect clusters. Using the standard terminology of strength of a cross-section, and moreover idealizing the short defect cluster intervals to be concentrated in discrete cross-sections, in the following called the weak cross-sections, it is assumed that

$$M_f = X + \min\{Y_1, \dots, Y_k\} \tag{1}$$

is the bending strength of the beam subject to constant bending moment over a span that contains k weak cross-sections. X is a random variable over the population of trees, while $Y_1, \dots$ are mutually independent zero mean random variables over the population of weak cross-sections. Thus the model assumes that bending faillure can only occur in one of the weak cross-sections. The bending strength of the ith weak cross-section, $X + Y_i$, has the variance $\text{Var}[X+Y_i] = \sigma_X^2+\sigma_Y^2$ where $\sigma_X^2 = \text{Var}[X]$ and where it is assumed that $\text{Var}[Y_i] = \sigma_Y^2$ is a constant independent of i. Since $\text{Cov}[X+Y_i, X+Y_j] = \sigma_X^2$ it follows that the correlation coefficient between the bending strength in any two weak cross-sections within the same beam becomes $\text{Corr}[X+Y_i, X+Y_j] = \sigma_X^2/(\sigma_X^2+\sigma_Y^2) = \rho$ for $i \neq j \leq k$. Under the assumption that X and Y are independent of k from beam to beam, this correlation structure is denoted as equicorrelation. Reported results show indications of such equicorrelation [3, 5].
Possibly this two-level hierarchical model is oversimplified. A crack may develop between any two adjacent weak cross-sections triggering a combined failure for a smaller bending moment than that required to cause failure in any of the two weak cross-sections separately.

This possibility is taken into account by the three-level hierarchical bending strength model

$$M_f = X + \min\{Y_1, \dots, Y_k, Y_1 + T_{12}, Y_2 + T_{21}, Y_2 + T_{23}, \dots, Y_{k-1} + T_{k-1\,k}, Y_k + T_{k\,k-1}\} \quad (2)$$

where $T_{12}, T_{21}, T_{23}, T_{32}, \dots, T_{k-1\,k}, T_{k\,k-1}$ are $2(k-1)$ identically distributed random variables of mean μ_T and standard deviation σ_T. Different dependencies between these random variables may be assumed, e.g. that they are all mutually independent, or that $T_{i\,i-1} = T_{i\,i+1}$ with independence otherwise.

Keeping to the two-level hierarchical model (1) as a working hypothesis, it is anticipated that the necessary experimental information to estimate the parameters of the model and to check its assumptions can be obtained by bending strength measurement of each single weak cross-section in the way described in Section 1. By cutting only the one of the beams in each of the pairs of twins into single weak cross-section test pieces, the other beam in each pair is kept for measuring the bending strength of the entire beam. Thus it will be possible experimentally to test the predictions of the series system model (1) as statistically inferred from the single weak cross-section measurements. Following this plan 26 beams were cut into 197 test pieces and tested to bending failure. The applied experimental technique of finger splicing the test piece in both ends to shafts of stronger wood turned out not to be particularly successful. Only 34% of the test pieces failed in the weak cross-sections while 42% of the failures took place in one or the other of the finger splices including1% in the shafts. The remaining 24 % of the failures were of a mixed type. Thus an extended model must be formulated to cope with this situation, remembering that a measurement of a finger splice bending strength by a bending test with constant moment in all three cross-sections is a measurement of a lower bound on the bending strength of the weak cross-section.

3 Hierarchical model for statistical inference by maximum likelihood principle

Treating the splice cross-sections as weak cross-sections the simple hierarchical model

$$Z = \min\{X + Y, R + \min\{S_1, S_2\}\} \quad (3)$$

is an obvious extension of (1) for the test piece bending strength (omitting indexing with respect to test piece number). The variables $R + S_1$, and $R + S_2$ are the random bending strengths of the two splice cross-sections, with the random variable R being the same for all test pieces cut from the same beam. All Y-variables and S-variables are assumed to be mutually independent and normally distributed with zero mean and standard deviations σ_Y, σ_S, respectively.

The full experimental information is taken into account in the likelihood function for the parameters $(x_1, r_1), \dots, (x_N, r_N)$ [sample of pairs of estimates of (X, R)] and σ_Y, σ_S given the $n_1 + \dots + n_N = 197$ observations of the pairs (Z, failure type). The observed failure types are denoted by $\mathcal{B}$ for bending failure in the weak cross-section, $\mathcal{F}$ for failure in the finger joints, and $\mathcal{BF}$ for a combined failure. This last type of failure information is on this level of modeling interpreted as the occurrence of the union event $\mathcal{F} \cup \mathcal{B}$, i.e. its is undecided whether $\mathcal{F}$ or $\mathcal{B}$ has occurred. Since

$$\frac{\partial}{\partial z} P(\{Z \le z\} \cap \mathcal{B} \,|\, X = x, R = r) = P(\min\{S_1, S_2\} > z - r) f_Y(z - x) \tag{4}$$

$$\frac{\partial}{\partial z} P(\{Z \le z\} \cap \mathcal{F} \,|\, X = x, R = r) = P(Y > z - x) f_{\min\{S_1,S_2\}}(z - r) \tag{5}$$

the likelihood function becomes

$$\begin{aligned} &L[(x_1, r_1) \ldots, (x_N, r_N); \sigma_Y, \sigma_S] \\ &= \prod_{i=1}^{N} \prod_{j=1}^{n_i} \Big\{ P(\min\{S_1, S_2\} > z_{ij} - r_i) f_Y(z_{ij} - x_i)[I_{\mathcal{B}}(i,j) + I_{\mathcal{BF}}(i,j)] \\ &+ P(Y > z_{ij} - x_i) f_{\min\{S_1,S_2\}}(z_{ij} - r_i)[I_{\mathcal{F}}(i,j) + I_{\mathcal{BF}}(i,j)] \Big\} \end{aligned} \tag{6}$$

where $I_{\mathcal{B}}(i,j)$, $I_{\mathcal{F}}(i,j)$, and $I_{\mathcal{BF}}(i,j)$ are the indicator functions for the events $\mathcal{B}$, $\mathcal{F}$, and $\mathcal{BF}$, respectively, observed in the bending test with the jth weak cross-section in the ith beam. With the normal distribution assumption the two factors in (3) to the indicator functions become

$$P(\min\{S_1, S_2\} > z_{ij} - r_i) f_Y(z_{ij} - x_i) = \frac{1}{\sigma_Y} \varphi\Big(\frac{z_{ij} - x_i}{\sigma_Y}\Big) \Phi\Big(\frac{r_i - z_{ij}}{\sigma_S}\Big)^2 \tag{7}$$

$$P(Y > z_{ij} - x_i) f_{\min\{S_1,S_2\}}(z_{ij} - r_i) = \frac{2}{\sigma_S} \Phi\Big(\frac{x_i - z_{ij}}{\sigma_Y}\Big) \Phi\Big(\frac{r_i - z_{ij}}{\sigma_S}\Big) \varphi\Big(\frac{r_i - z_{ij}}{\sigma_S}\Big) \tag{8}$$

respectively, where $\Phi(x) = \int_{-\infty}^{\infty} \varphi(u)\, du$, $\varphi(x) = \exp(x^2/2)/\sqrt{2\pi}$. For three of the beams all failure modes were observed to be of type $\mathcal{F}$. Then, obviously, the test results for these beams only contain weak information about X and Y except that the realizations of $X + Y$ are larger than the measured bending strengths. This means that the maximum likelihood estimation of the corresponding x-parameters is highly unstable. In fact, their values may formally be put to ∞ without it having essential effect on the estimation of the remaining parameters, that is, the factor $\Phi[(x_i - z_{ij})/\sigma_Y]$ becomes so close to 1 in the maximizing procedure that it might as well be put to 1. Similarly, for one of the beams all failure modes were observed to be of type $\mathcal{B}$. Then the corresponding r-parameters can formally be put to ∞. When running the maximization procedure the unstable parameters directly show up. For the actual sample of 26 beams, six x-parameters and three r-parameters turn out to be unstable. These cases are all with failures of type $\mathcal{F}$ and some few of type $\mathcal{BF}$, or all of type $\mathcal{B}$ and some few of type $\mathcal{BF}$, respectively. A single observation of a shaft failure has been classified as a failure of type $\mathcal{F}$. The obtained maximum likelihood estimates of the standard deviations are $\sigma_Y \approx 2.0$ kN and $\sigma_S \approx 1.7$ kN. The samples of 20 estimates of x-parameters and 23 r-parameters both fit reasonably well to normal distributions with mean values $\mu_X = 14.7$ kN, $\mu_R = 16.6$ kN and standard deviations $\sigma_X = 2.9$ kN, $\sigma_R = 1.6$ kN, respectively. The sample of (x, r) contains only 17 pairs from which the correlation coefficient estimate $\rho_{X,R} \approx 0.68$ is obtained.

The estimate of the equicorrelation coefficient $\rho = \sigma_X^2/(\sigma_X^2 + \sigma_Y^2)$ obtained from substituting the estimates of σ_X and σ_Y becomes 0.68. For comparison the Williamson [5] has estimated the value 0.58.

4 Predicted distribution of full beam strength

The number of weak cross-sections within the span of constant bending moment is a random variable K completely defined by its probability generating function function $\psi(x) = E[x^K]$. The distribution function of the beam bending strength M_f according to the hierarchical model (1) is then obtained in the two steps

$$F_{M_f}(z \mid K = k) = \frac{1}{\sigma_X} \int_{-\infty}^{\infty} \Big[1 - \Phi\Big(-\frac{z-x}{\sigma_Y}\Big)^k\Big] \varphi\Big(\frac{x-\mu_X}{\sigma_X}\Big)\, dx \tag{9}$$

$$F_{M_f}(z) = \sum_{k=0}^{\infty} F_{M_f}(z \mid k)\, p_k = 1 - \frac{1}{\sigma_X} \int_{-\infty}^{\infty} \psi\Big[\Phi\Big(\frac{x-z}{\sigma_Y}\Big)\Big] \varphi\Big(\frac{x-\mu_X}{\sigma_X}\Big)\, dx \tag{10}$$

As an example one might assume that K has a Poisson distribution with parameter λL where L is the span. Then $\psi(x) = e^{\lambda L(x-1)}$ so that (10) becomes

$$F_{M_f}(z \mid K = k) = 1 - \frac{1}{\sigma_X} \int_{-\infty}^{\infty} \exp\Big[-\lambda L \Phi\Big(-\frac{z-x}{\sigma_Y}\Big)\Big] \varphi\Big(\frac{x-\mu_X}{\sigma_X}\Big)\, dx \tag{11}$$

$$\to 1 - \exp\Big[-\lambda L \Phi\Big(-\frac{z-\mu_X}{\sigma_Y}\Big)\Big] \tag{12}$$

as $\sigma_X \to 0$. The limit distribution (12) was suggested as bending strength distribution for wood beams by Riberholt and Madsen (1979)[3].

5 Measured distribution of full beam strength

It turns out that the a reasonably good model distribution fit is obtained to the empirical distribution function of the sample of 197 test piece strengths. The estimated model distribution function is based on the hierarchical model (3) with the assumption of normal distribution of all the random variables in the model and maximum likelihood estimation

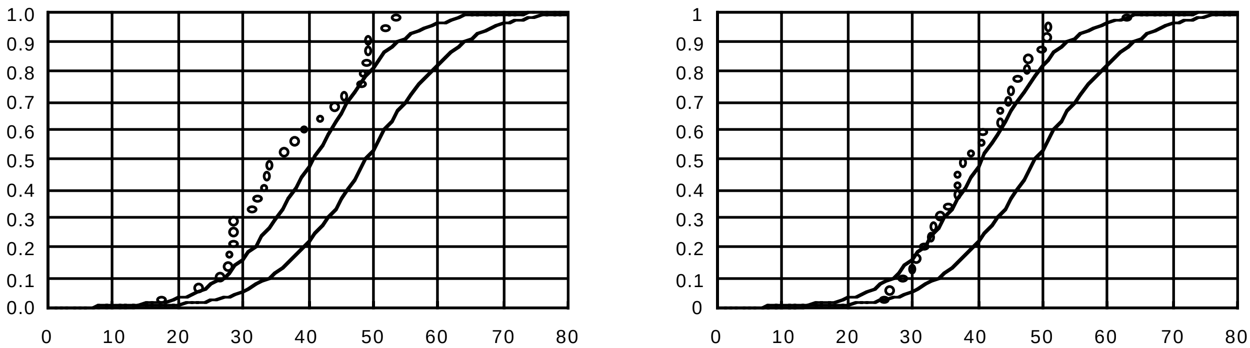

Figure 2: Left: Empirical distribution function for the bending strength data (in N/mm^2) of the 26 twin beams to the test piece subdivided beams plotted together with the estimated distribution function (10) obtained from the hierarchical model (left curve). Right: The same for the bending strength data of the 14 twin beam pairs. In both figures the right curve is the estimated distribution function (9) for $k = 1$ (normal distribution).

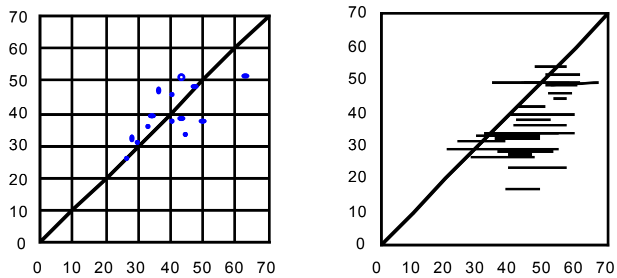

Figure 3: Left: Bending strength value pairs (in N/mm^2) for the 14 pairs of twin timber beams cut out from the same log. Right: Bending strength values for twin beams to the test piece specimens marked as horizontal line pieces. The end points of these line pieces mark the smallest and the largest measured bending strengths of the test pieces cut from the other twin member.

of the parameters. However, the decisive test of the applicability of the formulated combined hierarchical and series system model is a comparison of the bending strength distribution function (10) derived from the model with the empirical distribution functions obtained from independent bending tests of full beams. For this purpose the $N = 26$ twin beams to those cut into pieces and 14 extra pairs of full twin beams were tested to bending failure in a test arrangement as illustrated in Fig. 1. The details of these tests are described in [2, 4]. The obtained empirical distribution functions are shown in Fig. 2 together with the predicted theoretical distribution function. It is fair to say that the prediction is a success.

The 14 pairs of twin beams have been selected such that they judgmentally all contain $k = 6$ weak cross-sections (except one pair with 7 weak cross-sections). The strength pair values are plotted in Fig. 3 (left). According to the hierarchical model the correlation coefficient is

$$\begin{aligned}\rho_2 &= \frac{\mathrm{Var}[X] + \mathrm{Cov}[\min\{Y_{11},\dots,Y_{1k}\},\min\{Y_{21},\dots,Y_{2k}\}]}{\mathrm{Var}[X] + \mathrm{Var}[\min\{Y_{11},\dots,Y_{1k}\}]} \\ &= \frac{\mathrm{Var}[X] + \zeta_k \mathrm{Var}[Y]\mathrm{Corr}[\min\{Y_{11},\dots,Y_{1k}\},\min\{Y_{21},\dots,Y_{2k}\}]}{\mathrm{Var}[X] + \zeta_k \mathrm{Var}[Y]} \end{aligned} \tag{13}$$

where ζ_k is the variance reduction factor that can be calculated for any given distribution of Y. For the normal distribution it is $\zeta_k \approx 0.42$ for $k = 6$. Moreover, consistent with the hierarchical model, it is reasonable to assume that $\mathrm{Corr}[Y_{1i}, Y_{2j}] = \rho_{12}\,\delta_{ij}$ where ρ_{12} is a constant correlation coefficient. Under the normal distribution assumption it can then be shown that $\mathrm{Corr}[\min\{Y_{11},\dots,Y_{1k}\}, \min\{Y_{21},\dots,Y_{2k}\}] \geq -\mathrm{Corr}[\min\{Y_{11},\dots,Y_{1k}\}, \max\{Y_{11},\dots,Y_{1k}\}]$, where the right hand side is about 0.13 for $k = 6$. This implies that ρ_2 has a lower bound of about 0.81, given the estimates $\sigma_X = 2.9$ kN and $\sigma_Y = 2.0$ kN. If $\mathrm{Corr}[Y_{1i}, Y_{2j}] = 0$ for all i, j, the correlation coefficient becomes $\rho_2 = 0.83$. The estimate obtained from the 14 observed bending strength pairs is $\rho_2 \approx 0.68$. In consideration of the large statistical uncertainty of a correlation estimate when obtained from as small a sample size as 14, the violation of the lower bound 0.81 is not significant. In fact, for $\mathrm{Corr}[Y_{1i}, Y_{2i}] \leq 0$, simulation of a large number of estimates from independent samples of 14 pairs gives a standard deviation of the estimator of ρ_2 of about 0.1.

As a function of $\mathrm{Corr}[Y_{1i}, Y_{2i}]$ the correlation coefficient $\mathrm{Corr}[\min\{Y_{11},\dots,Y_{1k}\}, \min\{Y_{21}, \dots, Y_{2k}\}]$ is almost constant and equal to -0.13 for $-1 \leq \mathrm{Corr}[Y_{1i}, Y_{2i}] \leq -0.5$ and is slowly increasing to 0 for $\text{Corr}[Y_{1i}, Y_{2i}]$ increasing to 0. However, due to the large statistical uncertainty it cannot be concluded with large confidence that $\text{Corr}[Y_{1i}, Y_{2i}]$ is negative since (0.83-0.68)/0.1 = 1.5 is not an unlikely long distance in terms of the standard deviation of the correlation coefficient estimator. If $\rho_{12} < 0$, the physical explanation may be that at a weak cross-section there is a tendency of compensation for a below average strength in the one half of the log by an above average strength in the other half of the log.

6 Interaction between two adjacent weak cross-sections

The empirical distribution function plotted in Fig. 2 (left) may indicate that a part of the beams failed by combined failures with interaction between adjacent defect clusters. To investigate whether the effect of such interactions is significant a test series has been made at Trätek with spliced test pieces containing two adjacent defect clusters. The results have not yet been considered for statistical analysis on the basis of the three-level hierarchical model (2).

Note

The parameter estimates in the references [1, 2, 4] are based on a simpler hierarchical model than considered herein. The model does not take the complete experimental information from the single weak cross-section tests into account. It turns out that the more correct estimate of σ_X obtained herein is about 40% larger than the estimate obtained in [1]. The other parameters common to the two models do not change as much. One consequence is that the estimate of the equicorrelation coefficient changes from about 0.55 in [1] to about 0.68 herein.

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References

[1] Källsner, B., and Ditlevsen, O.: Lengthwise bending strength variation of structural timber. *IUFRO/S 5.02 Timber Engineering Meeting*, Sydney, Australia. Trätek. Rapport I 9412072, Stockholm, 1994.

[2] Källsner, B., Ditlevsen, O., and Salmela, K.: Experimental verification of a weak zone model for timber in bending. *IUFRO/S 5.02 Timber Engineering Meeting*, Copenhagen, June 1997.

[3] Riberholt, and H. Madsen, P. H.: *Strength distribution of timber structures, measured variation of the cross sectional strength of structural timber.* Structural Research Laboratory, Technical University of Denmark, Report No. R 114,1979.

[4] Salmela, K.: *Bending strength of timber at long-span testing* . Master thesis (in Swedish) at Building Materials, Department of Buildings and Building Services, Royal Institute of Technology, Stockholm, Sweden, 1997.

[5] Williamson J.A.: Statistical dependence of timber strength. *IUFRO/S 5.02 Timber Engineering Meeting*, Sydney, Australia, 1994.