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Random Field Interpolation Between Point by Point Measured Properties

Ove Ditlevsen
Department of Structural Engineering, Technical University of Denmark, Building 118, DK 2800 Denmark

Abstract Computational practicability imposes strong restrictions on the choice of a random field model for stochastic interpolation (kriging) between measured values of a material property, say, at a large set of points in a material body. With background in an actual example of a large number of cone tip resistance measurements in clay till it is demonstrated that pragmatic considerations lead to an almost unique mathematical structure of the model. The problem of pragmatism originates from the necessity of being able to invert a very large covariance matrix a large number of times. In order to appreciate this a general introduction to random field interpolation is given.

Introduction

Interpolation between point by point measured values of a spatially distributed property has been based on statistical principles for decades within the field of mining and mineral prospecting. In this field of mining geostatistics the method is called "kriging" alluding the works of D.G. Krige quoted in Journel and Huijbregts [1]. In a framework of reasoning very much different from the one considered in the following G. Matheron [2,3] has formulated the philosophy and the principles of geostatistics that includes the principles of kriging.

In modern reliability analysis in particular in geotechnical engineering [4] there is a need to make interpolation that includes measures of uncertainty of the interpolation. This paper first reconsiders the classical problem of interpolation and the difficulty of error estimation in the classical theory except under conditions that only rarely are satisfied in practice. The modern interpretation of the kriging procedure as the formulation of a conditional random field model is next introduces as a pragmatic alternative to the classical interpolation technique. Inherently it contains a probabilistic measure of the interpolation error. The random field method of interpolation as such has been considered and applied in several papers on engineering applications [4] but generally without a discussion of a proper interpretation of the method. Herein it is attempted to present the method with a linking to classical interpolation. This philosophical basis may ease the understanding of the method as a stochastic interpolation method which is not tied to some physical process

that behaves according to a random mechanism. It is merely a rational model of the uncertain engineering guess on interpolated values. It rests on the idea that continuity and differentiability principles make the variation among the measured values representative for what should be expected with respect to the uncertainty of the interpolation values.

Finally it is shown that pragmatic principles of computational practicability of random field interpolation in large regularly organized tables lead to a very narrow class of applicable correlation functions.

Classical interpolation

The solution to the problem of interpolation in a table (x_0,y_0), (x_1,y_1), ..., (x_{n-1},y_{n-1}) of values of the n times differentiable function $y = f(x)$ is in classical mathematical analysis given by Lagrange on the form

$$f(x) = y_0\, Q_0(x) + y_1\, Q_1(x) + \ldots + y_{n-1}\, Q_{n-1}(x) + R_n(x) \qquad (1)$$

in which $Q_i(x)$, $i = 0, 1, \ldots, n-1$, is the uniquely defined polynomial of (n–1)th degree that takes the value 1 for $x = x_i$ and the value 0 for $x = x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1}$. The Lagrangian remainder $R_n(x)$ is

$$R_n(x) = (x-x_0) \cdot \ldots \cdot (x-x_{n-1}) \frac{f^{(n)}(\zeta)}{n!} \qquad (2)$$

in which ζ is some number contained within the smallest interval I that contains $x, x_0, \ldots, x_{n-1}$. Interpolation to order $n-1$ then consists of using (1) with the remainder neglected. The error committed by the interpolation is bounded in absolute value by

$$|R_n(x)| \leq \frac{|(x-x_0)\cdot\ldots\cdot(x-x_{n-1})|}{n!} \max_{\zeta \in I} |f^{(n)}(\zeta)| \qquad (3)$$

Thus the error evaluation in classical interpolation is not offering a model of the error in terms of a random variable but only an upper bound that can be highly conservative. Moreover, the evaluation of the bound (3) requires that the nth derivative can be obtained. This, however, is in contrast to most situations in practice where interpolation is needed. Often only the table (x_0,y_0), (x_1,y_1), ..., (x_{n-1},y_{n-1}) is given and the values of the function f(x) are unknown between these points. This is the typical situation when the table is obtained by point by point measurement of some physical quantity that varies with x in an unknown way. The table may also be the result of a lengthy computation. In principle f(x) may be computed for any needed value of x but it may be more economical to accept even a considerable error in the evaluation of f(x) as obtained by interpolation between already computed values. The same may hold in the case of values obtained by measurements. In some situa-

tions it may even be impossible to measure intermediate values (as in the obvious case of x being the time but sometimes also in cases where x is a spatial coordinate). With only the table of points given the interpolation procedure is usually guided by a principle of simplicity perhaps supported by physical and geometrical arguments. To appreciate the philosophy of the principle of simplicity behind practical interpolation it is only needed to note that there is an infinite set of interpolation functions passing through the given points. Besides satisfying differentiability requirements to some specified order this set is obviously characterized by the property that the difference between any two functions of the set is a function with zero points at $x_0, x_1, \ldots, x_n$. In practice the choice of the interpolation function from the set is usually made in view of choosing an interpolation function that in itself and in all the derivatives up to the specified order has as small a variability as possible. Even though this principle of simplicity of the interpolation may be given a precise mathematical definition and it thereafter may be demonstrated that it leads to a unique choice of the interpolation function, it is still just an arbitrary principle that does not give any indication of the interpolation error.

A solution to this problem can be obtained by introducing an alternative but somewhat similar principle of simplicity. It is based on the principles of statistical reasoning.

Random field interpolation. Maximum likelihood principle of choice

Let $J(y_0,\ldots,y_{n-1})$ be the set of interpolation functions and consider the union

$$\mathcal{F} = \bigcup_{z\in\mathbb{R}^n} J(z) \tag{4}$$

in which $z = (z_0,\ldots,z_{n-1})$. In case it is only required that any interpolation function is n times continuously differentiable, then $\mathcal{F}$ is the class of n times continuously differentiable functions. Define a probability measure over $\mathcal{F}$ that depends on a number of parameters $\theta_1, \ldots, \theta_q$. For each value set of $\theta_1, \ldots, \theta_q$ we then have a random field over the range of values of x. Let the probability measure have such properties that there is a probability density at the set $J(z)$ for each $z \in \mathbb{R}^n$ in the sense of being the unique limit

$$\lim_{d\to 0}\left\{\frac{P\left[\bigcup_{\zeta\in\mathcal{N}(z)} J(\zeta)\,\middle|\,\theta_1,\ldots,\theta_q\right]}{\mathrm{Vol}[\mathcal{N}(z)]}\right\} \tag{5}$$

where d = diameter of $\mathcal{N}(z)$, $\mathcal{N}(z)$ is an arbitrary neighborhood of z with volume $\mathrm{Vol}[\mathcal{N}(z)]$, and $P[\cdot\,|\,\theta_1,\ldots,\theta_q]$ is the probability measure.

The principle of simplicity now concerns a principle about how to choose the values of the parameters $\theta_1, \ldots, \theta_q$ that fix the probability

measure on $\mathcal{F}$. Instead of the deterministic principle of least variability it is reasonable to choose $\theta_1, \ldots, \theta_q$ so that the probability density has its maximum for $\mathbf{z} = \mathbf{y} = (y_0, y_1, \ldots, y_{n-1})$, that is, at the actual set of interpolation functions. This is the well–known principle of maximum likelihood estimation in the theory of mathematical statistics. When the parameter values are chosen for example according to the principle of maximum likelihood, the probability measure on $\mathcal{F}$ induces a probability measure in the relevant set of interpolation functions $J(\mathbf{y})$ simply as the conditional probability measure given $J(\mathbf{y})$, that is, given $\mathbf{z} = \mathbf{y}$. Thus a conditional random field is defined that possesses the desired properties of expressing the interpolated value at x of the unknown function as a random variable. At the points of the given table $(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$ the random variable degenerate to be atomic, but at a point x different from the points of the table it gets a mean and a standard deviation that depend on x. The detailed nature of this dependency is laid down in the mathematical structure of the probability measure introduced in $\mathcal{F}$.

From an operational point of view the most attractive probability measure to choose is the Gaussian measure. Let the Gaussian measure on $\mathcal{F}$ have mean value function $\mu(x)$, $x \in \mathbb{R}$, and covariance function $c(\xi,\eta)$, $\xi, \eta \in \mathbb{R}$. Then the conditional measure on $J(\mathbf{y})$ is Gaussian with mean value function given by the linear regression

$$E[Y(x)\,|\,\mathbf{z}=\mathbf{y}] = \mu(x)+[c(x,x_0)\ldots c(x,x_{n-1})]\{c(x_i,x_j)\}^{-1}\begin{bmatrix} y_0 - \mu(x_0) \\ \vdots \\ y_{n-1} - \mu(x_{n-1}) \end{bmatrix} \quad (6)$$

and covariance function given by the residual covariance function corresponding to the linear regression. It is

$$\mathrm{Cov}[Y(\xi),Y(\eta)\,|\,\mathbf{z}=\mathbf{y}] = c(\xi,\eta) - [c(\xi,x_0)\ldots c(\xi,x_{n-1})]\{c(x_i,x_j)\}^{-1}\begin{bmatrix} c(\eta,x_0) \\ \vdots \\ c(\eta,x_{n-1}) \end{bmatrix} \quad (7)$$

Comparison with classical interpolation

The conditional mean (6), in particular, is written out explicitly in order to display an interpretation of the mean value function $\mu(x)$ and the function $c(\xi,\eta)$ that relates them to usual deterministic interpolation practice. In fact, if (1) and (6) are compared, it is seen that (6) is obtained from (1) if the polynomial $Q_i(x)$, $i = 0, \ldots, n-1$, is replaced by the linear combination

$$a_{oi}\, c(x,x_0) + a_{1i}\, c(x,x_1) + \ldots + a_{(n-1)i}\, c(x,x_{n-1}) \quad (8)$$

where the coefficients a_{oi}, ..., $a_{(n-1)i}$ are the elements of the ith column in the inverse to the covariance matrix $\{c(x_i,x_j)\}$. It is seen that this function like $Q_i(x)$ is 1 for $x = x_i$ and 0 for $x = x_o$, ..., x_{i-1}, x_{i+1}, ..., x_{n-1}. Indeed, if (8) is identified with $Q_i(x)$ for $i = 0, ..., n-1$, the functions $c(x,x_o)$, ..., $c(x,x_{n-1})$ are uniquely determined by their values at x_o, x_1, ..., x_{n-1} as

$$c(x,x_i) = c(x_o,x_i)\, Q_o(x) + ... + c(x_{n-1},x_i)\, Q_{n-1}(x) \tag{9}$$

As function of x this is a valid covariance, that is, there exists a non–negative definite function $c(\xi,\eta)$ such that (9) is obtained for $(\xi,\eta) = (x,x_i)$. This is seen directly by computing the covariance function of the random field

$$Z_o\, Q_o(x) + ... + Z_{n-1}\, Q_{n-1}(x) \tag{10}$$

in which $(Z_o,...,Z_{n-1})$ is a random vector with covariance matrix $\{c(x_i,x_j)\}$. The covariance function is

$$c(\xi,\eta) = [Q_o(\xi) \; ... \; Q_{n-1}(\xi)]\; \{c(x_i,x_j)\} \begin{bmatrix} Q_o(\eta) \\ \vdots \\ Q_{n-1}(\eta) \end{bmatrix} \tag{11}$$

which gives (9) for $\xi = x$ and $\eta = x_i$.

It follows from this that (1) is a special case of (6). The remainder becomes replaced by the term

$$\mu(x) - [c(x,x_o) \; ... \; c(x,x_{n-1})]\; \{c(x_i,x_j)\}^{-1} \begin{bmatrix} \mu(x_o) \\ \vdots \\ \mu(x_{n-1}) \end{bmatrix} \tag{12}$$

plus a Gaussian zero mean random field with covariance function given by (7). This interpretation of the mean value function $\mu(x)$ and the covariance function $c(\xi,\eta)$ of the random field as essentially being interpolation functions makes it easier to appreciate the consequence of a specific mathematical form of the covariance function. For example, if the general appearance of the values in the table suggests the choice of a homogeneous Gaussian field, the choice of a correlation function like $\exp[-\alpha|\,\xi-\eta\,|]$ implies that the mean interpolation function given by (6) will be non–differentiable at $x = x_o$, ..., x_{n-1} while a correlation function like

$\exp[-\beta(\xi-\eta)^2]$ will give a differentiable interpolation function also at $x = x_0, \ldots, x_{n-1}$. In fact, the sample functions corresponding to the first correlation function will with probability 1 be continuous but not differentiable at any point while the second correlation function corresponds to sample functions that are differentiable of any order.

Example 1 Consider the case where the points $x_0, \ldots, x_{n-1}$ are equidistant with $x_{i+1} - x_i = h$. Let the random interpolation field on the line be a homogeneous Gaussian field with mean μ and covariance function

$$c(\xi,\eta) = \begin{cases} \left[1 - \frac{|\xi-\eta|}{h}\right] \sigma^2 & \text{for } |\xi-\eta| \le h \\ 0 \text{ otherwise} \end{cases} \tag{13}$$

Then (6) gives

$$E[Y(x)\,|\,z=y] = \frac{1}{h}\,[(x_1-x)\,y_0+(x-x_0)\,y_1] \tag{14}$$

for $x_0 \le x \le x_1$. This is equivalent with linear interpolation in the mean. The covariance function (7) becomes

$$\mathrm{Cov}[Y(\xi),Y(\eta)\,|\,z=y] =$$

$$\begin{cases} \left[\frac{\sigma}{h}\right]^2 [h^2-h|\xi-\eta|-(x_1-\xi)(x_1-\eta)-(x_0-\xi)(x_0-\eta)] & \text{for } |\xi-\eta| \le h \\ 0 \text{ otherwise} \end{cases} \tag{15}$$

from which the standard deviation

$$D[Y(x)\,|\,z=y] = \frac{\sigma}{h}\sqrt{2(x-x_0)(x_1-x)} \tag{16}$$

is obtained. It is interesting to compare this measure of the interpolation error with the Lagrangian remainder obtained from (2) for $n = 2$. It is seen that (16) varies like the square root of $R_2(x)$, that is, it predicts larger errors close to x_0 or x_1 than $R_2(x)$ does.

While the interpolation functions in the classical procedure are broken at the points of the table implying that no way is indicated of reasonable extrapolation outside the range from x_0 to x_{n-1}, the homogeneous random field procedure represents a solution to both the interpolation and the extrapolation problem in terms of a set of sample functions defined everywhere. The points of the table only play the role that all the sample functions of the set pass through the points of the table. The homogeneity of the field implies that the variability of the points within

the range from x_0 to x_{n-1} is reflected in the field description outside this range. Specifically we have for $x \leq x_0$

$$E[Y(x)|z=y] = \begin{cases} \mu & \text{for } x \leq x_0 - h \\ \frac{1}{h}[(x_0-x)\ \mu+(x-x_0+h)\ y_0] & \text{for } x_0 - h < x \leq x_0 \end{cases} \tag{17}$$

and similarly for $x > x_{n-1}$. The standard deviation is

$$D[Y(x)|z=y] = \begin{cases} \sigma & \text{for } x \leq x_0 - h \\ \frac{\sigma}{h}\sqrt{(x_0-x)(x-x_0+2h)} & \text{for } x_0 - h < x \leq x_0 \end{cases} \tag{18}$$

and similarly for $x > x_{n-1}$. Outside the interval from $x_0 - h$ to $x_{n-1} + h$ the field is simply given as the homogeneous field. The variation among $y_0, \ldots, y_{n-1}$ determines the values of μ and σ^2 as the well–known maximum likelihood estimates

$$\bar{y} = \frac{1}{n}\sum_{i=0}^{n-1} y_i \quad \text{and} \quad s^2 = \frac{1}{n}\sum_{i=0}^{n-1}(y_i-\bar{y})^2 \tag{19}$$

corresponding to a Gaussian sample of independent outcomes $y_0, \ldots, y_{n-1}$.

Bayesian principle of choice

The maximum likelihood principle leads to a specific choice of the distributional parameters $\theta_1, \ldots, \theta_q$, that is, it leaves no room for doubt about what values to choose. Such doubt can be included in the stochastic interpolation procedure by assigning not just the distribution family to the interpolation problem but also assigning a joint probability distribution to $\boldsymbol{\theta} = (\theta_1,\ldots,\theta_q)$. The mathematical technique then becomes exactly the same as in the Bayesian statistical method: The joint probability density of the parameter vector $\boldsymbol{\theta}$ and any vector of field random variables that includes the random variables $Y(x_0), \ldots, Y(x_{n-1})$ as a subvector is obtained as the product of the given relevant conditional probability density of the field random variables given the parameters and the assigned joint density of the parameter vector (the prior). With only the interpolation function set $J(\mathbf{y})$ being relevant, the conditional joint density of the parameter vector and the field random variables given $(Y(x_0),\ldots,Y(x_{n-1})) = (y_0,\ldots,y_{n-1})$ is obtained except for proportionality

by only considering the joint density at points for which $Y(x_0)$, ..., $Y(x_{n-1})$ are fixed at y_0, ..., y_{n-1} respectively. Integration over $\mathbb{R}^q$ with respect to the parameters then finally gives the joint distribution of the field random variables (the predictive distribution). The family of all predictive distributions obtained in this way defines the random field of interpolation and extrapolation functions such that it includes the statistical uncertainty effect of the size of the given table (x_0,y_0), ..., (x_{n-1},y_{n-1}).

Example 2 The parameter vector (μ,σ) in example 1 may be considered as an outcome of the random vector (M,Σ). Assuming a non–informative prior of $(M,\log \Sigma) \in \mathbb{R}^2$ (degenerate uniform density over $\mathbb{R}^2$), the special random field model of example 1 leads to the well–known Bayesian standard results in Gaussian statistics. For example, in the interpolation case the predictive distribution of

$$\frac{Y(x) - \frac{1}{h}\,[(x_1-x)y_0+(x-x_0)y_1]}{\frac{s}{h}\,\sqrt{2(x-x_0)(x_1-x)}}\,\sqrt{1-\frac{1}{n}} \tag{20}$$

for $x \in]x_0,x_1[$ is the t–distribution with $n-1$ degrees of freedom.

Computational practicability

Both for the maximum likelihood principle and the Bayesian principle of stochastic interpolation on the basis of a Gaussian field assumption the governing mathematical function is the joint Gaussian density

$$f_{\mathbf{Y}}(y_0,\ldots,y_{n-1};\mu,\sigma,\mathbf{P}) \propto \frac{1}{\sqrt{\det(\mathbf{P})}}\,\frac{1}{\sigma^n}\exp\left[-\frac{1}{2\sigma^2}\,(\mathbf{y}-\mu\,\mathbf{e})'\,\mathbf{P}^{-1}(\mathbf{y}-\mu\,\mathbf{e})\right] \tag{21}$$

written here for the case of a homogeneous field with mean μ, standard deviation σ, and a correlation function that defines the correlation matrix $\mathbf{P}$ of the random vector $(Y(x_0),\ldots,Y(x_{n-1}))$. The correlation matrix $\mathbf{P}$ contains in its elements the unknown parameters of the correlation function. As function of μ, σ and the correlation parameters the right side of (21) defines the likelihood function. Generally neither $\mathbf{P}^{-1}$ nor the determinant $\det(\mathbf{P})$ can be expressed explicitly in terms of the correlation parameters. Thus in order to maximize the likelihood with respect to the parameters both $\mathbf{P}^{-1}$ and $\det(\mathbf{P})$ must be evaluated several times during an iteration procedure.

The Bayesian principle of stochastic interpolation has particular relevance for structural reliability evaluations of structures with carrying capacities that depend on the strength variation throughout a material

body. For example, a direct foundation on saturated clay may fail due to undrained failure extended over a part of the clay body. The evaluation of the failure probability corresponding to a specific rupture figure requires a specifically weighted integration of the undrained shear strength across the rupture figure. If the undrained shear strength is measured only at a finite number of points in the clay body, it is required to make interpolation and or extrapolation in order to obtain the value of the integrand in any relevant point of the body. Irrespective of whether the maximum likelihood principle is applied or whether the Bayesian principle is applied in order to properly take statistical uncertainty into account, it is required that $\mathbf{P}^{-1}$ be computed iteratively for different values of the correlation parameters. For example, for the Bayesian principle of choice this is the case when the reliability is evaluated by a first or second order reliability method (FORM or SORM) in a space of basic variables that among its dimensions includes the correlation parameters. In the search for the most central limit state point by some gradient method, say, the inverse $\mathbf{P}^{-1}$ has to be computed iteratively several times for different parameter values.

From this discussion it follows that computational practicability put limits to the order of $\mathbf{P}$ or it requires that the mathematical structure of $\mathbf{P}$ is such that $\mathbf{P}$ can be inverted analytically or such that $\mathbf{P}^{-1}$ can be obtained in terms of matrices of considerably lower order than the order of $\mathbf{P}$. A possibility of breaking down to lower order is present in case of what here is called factorized correlation structure:

Let $\mathbf{P}$ and $\mathbf{Q} = \{q_{ij}\}_{mm}$ be correlation matrices. Then the matrix defined as

$$[\mathbf{Q}] \circ \mathbf{P} \equiv \begin{bmatrix} q_{11}\mathbf{P} & q_{12}\mathbf{P} & \dots \\ q_{21}\mathbf{P} & q_{22}\mathbf{P} & \dots \\ \dots & & q_{mm}\mathbf{P} \end{bmatrix} \tag{22}$$

is a correlation matrix. The proof is as follows: Let $\mathbf{x}' = [\mathbf{x}_1' \dots \mathbf{x}_m']$ be an arbitrary vector and let $\mathbf{A}$ be an orthogonal matrix such that $\mathbf{A}'\mathbf{P}\mathbf{A} = \mathbf{\Lambda} = [\lambda_1 \dots \lambda_n]$ where $\lambda_1, \dots, \lambda_n$ are the non–negative eigenvalues of $\mathbf{P}$. Then

$$\mathbf{x}'([\mathbf{Q}] \circ \mathbf{P})\,\mathbf{x} = \sum_{i=1}^{m}\sum_{j=1}^{m} q_{ij}\,\mathbf{x}_i'\,\mathbf{P}\,\mathbf{x}_j = \sum_{i=1}^{m}\sum_{j=1}^{m} q_{ij}\,\mathbf{y}_i'\,\mathbf{\Lambda}\,\mathbf{y}_j =$$

$$\lambda_1 \sum_{i=1}^{m}\sum_{j=1}^{m} q_{ij}\,y_{ij}\,y_{j1} + \dots + \lambda_n \sum_{i=1}^{m}\sum_{j=1}^{m} q_{ij}\,y_{in}\,y_{jn} \tag{23}$$

in which $\mathbf{y}_i = \mathbf{A}\,\mathbf{x}_i$, $i = 1, \dots, m$. Since $\mathbf{Q}$ is a correlation matrix, the right

hand side of (23) is non–negative. Thus it follows that $[\mathbf{Q}] \circ \mathbf{P}$ is a correlation matrix.

It follows directly by the multiplication test that if $\mathbf{P}$ and $\mathbf{Q}$ are both regular, then

$$([\mathbf{Q}] \circ \mathbf{P})^{-1} = [\mathbf{Q}^{-1}] \circ \mathbf{P}^{-1} \tag{24}$$

By application of simple row operations first diagonalizing $\mathbf{P}$ to $\mathbf{D}$ in all places in (22) to obtain

$$\begin{bmatrix} q_{11}\mathbf{D} & q_{12}\mathbf{D} & \cdots \\ q_{21}\mathbf{D} & q_{22}\mathbf{D} & \cdots \\ \cdots & & \end{bmatrix} \tag{25}$$

and next diagonalizing (25) by exactly those row operations that diagonalize $\mathbf{Q}$, it follows that

$$\det([\mathbf{Q}] \circ \mathbf{P}) = [\det(\mathbf{Q})]^n [\det(\mathbf{P})]^m$$

Isotropic factorized correlation structure in the plane

Assume that the table of points corresponds to points in the plane arranged in a square mesh of equidistant points in both directions with mesh width L. Let there be k points in the first direction and ℓ points in the second direction in total giving $k\ell$ points with given values of the otherwise unknown function. The field random variables Y_{11}, Y_{21}, ..., Y_{k1}, Y_{12}, Y_{22}, ..., Y_{k2}, ..., $Y_{1\ell}$, $Y_{2\ell}$, ..., $Y_{k\ell}$ corresponding to the points (i,j) of the mesh are collected in a vector of dimension $k\ell$ and in the order as indicated. Then it is easily shown that the correlation matrix of order $k\ell$ for an arbitrary choice of $\mathbf{L}$ has the factorized structure as in (22) if and only if the random field is correlation homogeneous and the correlation function of the field can be written as the product of a correlation function solely of the coordinate along the first mesh direction and a correlation function solely of the coordinate along the second direction. In the class of such product correlation functions the correlation functions that correspond to isotropic fields are uniquely given as

$$\exp[-\beta r^2] \tag{27}$$

where r is the distance between the two points at which the random variables are considered and β is a free positive parameter. Thus the assumption of isotropy of the interpolation field and the requirement of computational practicability for large $k\ell$ make it almost a "must" to adopt the correlation function defined by (27). Writing (27) for r = L as the correlation coefficient κ, the correlation matrix of the vector of Y–values becomes $[S_1] \circ S_2$ where

$$S_i = \begin{bmatrix} 1 & \kappa & \kappa^4 & \dots & \kappa^{(\nu-1)^2} \\ \kappa & 1 & \kappa & \dots & \kappa^{(\nu-2)^2} \\ \dots & & & & \end{bmatrix} \tag{28}$$

$i = 1$ ($\nu = k$), $i = 2$ ($\nu = \ell$). These considerations generalize directly to 3 dimensions for a spatial equidistant mesh of $k\ell m$ points. Then the correlation matrix becomes $[[S_1] \circ S_2] \circ S_3$ with each of the matrices S_1, S_2, S_3 of form as in (28) in case of spatial isotropy.

For a soil body it may be reasonable to have isotropy in any horizontal plane but not necessarily in the 3–dimensional space. Then S_3 may be any other correlation matrix obtained from the correlation function in the vertical direction. When soil strength measurements are made for example by the socalled CPT method (Cone Penetration Test), the distance h between measurement points is much smaller in the vertical direction than the mesh width L in the horizontal direction. Then S_3 can be of impracticable high order. However, there is at least one case of such a matrix S corresponding to a homogeneous field for which S^{-1} is known explicitly. This case is

$$S = \begin{bmatrix} 1 & \rho & \rho^2 & \dots & \rho^m \\ \rho & 1 & \rho & \dots & \rho^{m-1} \\ \dots & & & & \end{bmatrix}; \quad S^{-1} = \frac{1}{1-\rho^2} \begin{bmatrix} 1 & -\rho & 0 & \dots & 0 \\ -\rho & 1+\rho^2 & -\rho & \dots & 0 \\ \dots & & & & \end{bmatrix} \tag{29}$$

which corresponds to a homogeneous Markov field on the line with $\rho = \exp[-\alpha h]$ where α is a free positive parameter.

This homogeneous Gaussian 3–dimensional field interpolation model with (27) defining the horizontal correlation properties and with Markov properties in the vertical direction has recently been used in a reliability study of the anchor blocks for the future suspension bridge across the eastern channel of Storebælt in Denmark [5]. The size of the interpolation problem in that investigation was $k = 6$, $\ell = 10$, $m = 150$ giving a table of 9000 points. Without the unique structuring of the random field correlation given here, the likelihood analysis would have been totally impracticable.

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