Elasto-plastic frame under horizontal and vertical Gaussian excitation

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ABSTRACT: Taking geometric non-linearity into account an oscillator of the form as a portal frame with a rigid traverse and with ideal-elastic-ideal-plastic clamped-in columns behaves under horizontal excitation as an ideal-elastic-hardening/softening-plastic oscillator given that the columns carry a tension/compression axial force. Assuming that the horizontal excitation of the traverse is Gaussian white noise, statistics related to the plastic displacement response are determined by use of simulation based on the Slepian model process method combined with envelope excursion properties. Besides giving physical insight the method gives good approximations to results obtained by slow direct simulation of the total response. Moreover, the influence of a randomly varying axial column force is investigated by direct response simulation. This case corresponds to parametric excitation as generated by the vertical acceleration component in a quake.

1 INTRODUCTION

Slepian model process technique is useful for obtaining approximate distributions related to the plastic displacement response of a certain class of Gaussian excited elasto-plastic oscillators (Ditlevsen & Bognár 1993, Ditlevsen & Randrup-Thomsen 1996, Ditlevsen & Tarp-Johansen 1997, Randrup-Thomsen & Ditlevsen 1997). These oscillators are characterized by the property of being linear for external excitation responses that are inside a given more or less wide neighbourhood of the zero response. Such an oscillator is in the following called an EPO while the oscillator obtained from the EPO by extending the linearity neighbourhood to the entire response space is denoted as the corresponding ALO (Associated Linear Oscillator). The basis for applying the Slepian model process technique on the EPO is that the ALO for a stationary external Gaussian excitation has a stationary Gaussian response. Then the stochastic response sample function can be described by a Slepian model process in the vicinity of an outcrossing through the boundary of the linearity domain of the EPO. A Slepian model process is the sum of a non-Gaussian random variable determined process (the linear regression part) and a nonstationary Gaussian process (the residual process part) that in the vicinity of the outcrossing point is almost vanishing. Assuming the further simplification that the excitation is white noise, the residual process part becomes sufficiently simple to obtain closed form expressions for two different fundamental conditional distributions for the local extreme of the ALO response outside the boundary directly after an outcrossing through the boundary. These two conditional distributions are used to obtain an approximate description of the stochastic behavior of the plastic response of the EPO after the outcrossing to the plastic domain and until the return to the linear domain with zero velocity.

In the mathematical model of a random process excited structure the horizontal and the vertical excitation component enter quite differently. The horizontal component acts as an external excitation while the vertical component gives parametric excitation. The characterization of a structural system as being linear usually refers to the property that the response is a linear mapping of the external excitation. However, in general such linearity is not preserved for parametric excitation. This means, that Gaussianity of the excitation of a linear structure is only preserved if the excitation is solely external. Thus the ALO corresponding to a parametrically excited structure of EPO type does no longer have a Gaussian response. This raises a problem about the applicability of the Slepian model process technique to obtain the distributions of the plastic displacement process in the presence of Gaussian parametric excitation on top of the Gaussian white noise external excitation.

Herein we will study a portal frame of EPO type with a rigid and heavy traverse excited horizontally by a stationary Gaussian white noise and vertically and identically at both columns by the sum of the static load from the weight of the traverse and a zero mean stationary non-white Gaussian process. The vertical random process load on a shear frame structure with rigid traverses excites the structure through the geometric nonlinearity imposed on the column connections by the horizontal displacements of the traverses. This geometric nonlinearity gives a magnification of the horizontal displacement caused by the horizontal excitation component. When the structure is of EPO type the response will become nonstationary due to the plastic drift. Without taking the geometric nonlinearity into account the response will get stationary increments. If the geometric nonlinearity is included, the response gets nonstationary increments with these having a dominating tendency to increase by time.

If we restrict the vertical load to be static and below the stability load of the columns, the response of the ALO will be Gaussian and the Slepian model process technique is applicable in principle. However, it needs to be documented that reasonably accurate distribution results can be obtained in the presence of the geometric nonlinearity that causes the plastic increment distributions to change in dependence of the past history of plastic displacement occurrences. The paper is mainly devoted to this excercise. To check the obtained results the same examples are investigated empirically by direct simulation of a large number of response sample functions that subsequently are analysed statistically. A possible vertical random process excitation is solely taken into account by an approximately equivivalent static value. Finally, though, some examples with vertical random excitations are simulated for comparison.

2 DIMENSIONLESS EPO

The equation of motion of the elasto-plastic column supported rigid and heavy traverse is derived in Appendices 1 and 2. The traverse is subject to a horizontal stationary Gaussian white noise excitation and a vertical stationary random process excitation superposed on a static vertical load. In the elastic domain the equation can be given the following dimensionless form:

$$\ddot{X}(\tau)+2\alpha\dot{X}(\tau)+\frac{g(\lambda)}{g(\lambda_0)}(1+\alpha^2)[X(\tau)-X_{i-1}]=W(\tau) \tag{1}$$

The function $g(\lambda)$ is defined by (36), and $\lambda \geq -\pi$ (tension for $\lambda > 0$, compression for $\lambda < 0$), is a parameter that represents the total vertical load (λ^2/π^2 is the absolute value of the vertical load expressed as a fraction of the Euler stability load $\pi^2 EI/L^2$ for the columns). The dimensionless displacement $X(\tau)$ and the dimensionless time τ are defined such that the stationary response of the corresponding dimensionless ALO gets unit standard deviation and free damped vibrations of period 2π when the vertical load parameter λ is kept constant $= \lambda_0$. The damping parameter α in (1) is $\alpha = \zeta/\sqrt{g(\lambda_0)-\zeta^2}$ where ζ is the viscous damping ratio of the ALO. Moreover $X_0 = 0$ and

$$X_i = \frac{X_{pi}}{r(\lambda)}, \quad i \in \mathbb{N}, \quad r(\lambda) = \left(1 - \frac{\lambda\,|\lambda|}{12\,g(\lambda)}\right)^{-1} \tag{2}$$

where X_{pi} is the dimensionless accumulated plastic displacement after the ith visit to the plastic domain. The external excitation $W(\tau)$ is Gaussian white noise with intensity $4\alpha(1+\alpha^2)$, that is, with $\mathrm{Cov}[W(\tau_1), W(\tau_2)] = 4\alpha(1+\alpha^2)\delta(\tau_1-\tau_2)$. In the plastic domain the differential equation is

$$\frac{\lambda\,|\lambda|}{12\,g(\lambda_0)}(1+\alpha^2)[X(\tau)+\mathrm{sign}(\dot{X})\frac{12\,A}{\lambda\,|\lambda|}]$$

$$+\ddot{X}(\tau)+2\alpha\dot{X}(\tau)=W(\tau) \;\text{ for} \tag{3}$$

$$X\,\mathrm{sign}(\dot{X}) > X_{p\,i-1}\,\mathrm{sign}(\dot{X}) + h(\lambda)A \tag{4}$$

where $h(\lambda) = 1/[g(\lambda) - \lambda\,|\lambda|/12]$ is the function (38) and A is the dimensionless constant (44) proportional to the ratio of the column yield moment and the square root of the intensity of the external white noise excitation. The dimensionless displacement-force diagram is shown in Fig. 1 for

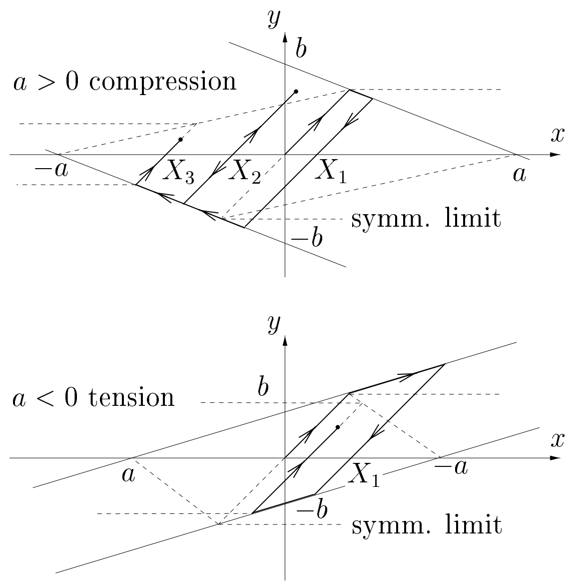

Figure 1. Illustration of clump definition as interpreted on the idealized dimensionless displacement-restoring force diagram corresponding to constant normal force $\lambda = \lambda_0$ in the frame columns.

the constant value λ_0 of the column force parameter λ. The diagram is idealized by not taking into account that the frame height decreases for increasing displacement. Thus the diagram is as the previous equations only valid in the domain of small displacements. Nevertheless, the diagram is well suited for illustrating the definition of what in the following is called the symmetrized elastic domain and also the definition used herein for a clump of plastic displacements. A crossing into the plastic domain marks the start of a clump of plastic displacements if the local extreme of the response directly before the outcrossing is in the local symmetrized elastic domain. The clump terminates when for the first time after the start an incrossing to the elastic domain is followed by a local extreme which is inside the local symmetrized elastic domain.

3 PLASTIC DISPLACEMENTS FOR CONSTANT VERTICAL LOAD

3.1 *Recursive relations*

In the following we will use the notation x for the dimensionless displacement and y for the dimensionless restoring force, and we define

$$a = \frac{r(\lambda_0)A}{[1-r(\lambda_0)]g(\lambda_0)}, \quad b = \frac{A(1+\alpha^2)}{g(\lambda_0)} \tag{5}$$

and $c = 1+\alpha^2$. Setting $r = r(\lambda_0)$ the equations of the displacement-force relation become:

Elastic part:

$$y = c(x - X) \tag{6}$$

$$-1 - \frac{X}{a} < \frac{x - X}{a(1-r)} < 1 - \frac{X}{a} \tag{7}$$

Upper/lower plasticity bound:

$$\frac{x}{a} + \frac{y}{b} = \pm 1 \tag{8}$$

Upper/lower symmetrized elasticity bound:

$$\frac{x}{a} - \frac{(2-r)y}{rb} = \mp 1 \tag{9}$$

where rX is the net accumulated plastic displacement. The symmetrized elasticity limits are

$$-1 + \left|\frac{X}{a}\right| < \frac{x - X}{a(1-r)} < 1 - \left|\frac{X}{a}\right| \tag{10}$$

Let $rX_0 = 0, rX_1, rX_2, \ldots$ be the sequence of accumulated net plastic displacements after each return to the elastic domain, and let $\xi_i = X_i/a$ where $-1 < \xi_i < 1$. Moreover, let M^2 be the square of the first displacement extreme of the ALO after upcrossing of the upper elasticity limit $(a - X)(1 - r)$ or downcrossing of the lower elasticity limit $-(a + X)(1 - r)$. Under the approximation that the excess elastic energy $c(M^2 - u^2)/2$ for $u = (a - X)(1 - r)$ or $u = -(a + X)(1 - r)$ at the elasticity limit passage can be put equal to the energy absorbed by the EPO, Karnopp-Scharton (Karnopp & Scharton 1966), the following recursive relation is obtained:

$$\xi_{i+1} = \frac{\dot{x}}{|\dot{x}|}\Big(1 - \sqrt{r^{-1}[(\xi_i - \dot{x}/|\dot{x}|)^2 - (1-r)^{-1}(M/a)^2]}\Big) \tag{11}$$

3.2 *Clump definition*

Per definition a clump of successive plastic displacements starts whenever a pair of successive local extreme displacements has the property that the first is inside and the second is outside the symmetrized elastic domain, Fig. 1. After its start the clump terminates the first time there is a pair of local extreme displacements with the property that the first is outside and the second is inside the symmetrized elastic domain. This definition implies that clumps without plastic displacements may occur with positive probability.

Let $ra\eta_j$ be the accumulated plastic displacement at the end of the jth clump of plastic displacements. The $(j+1)$th clump then starts when two successive ALO displacement extremes have the property that the first extreme is inside and the second extreme is outside the interval bounds $\pm u_j = \pm(1 - |\eta_j|)a(1 - r)$.

3.3 *Slepian model distributions*

Referring to (Ditlevsen & Bognár 1993, Ditlevsen & Tarp-Johansen 1997) the plastic displacement clump simulation uses the following two Slepian model based conditional distributions of $M > u$:

$$x \ge \frac{u^2}{2}: \quad P(M > x \,|\, -u < X(-\pi) < u) =$$

$$\frac{\varphi\Big(\frac{x}{\sqrt{k}}\Big)\Phi\Big(\frac{\xi - \mu x}{\sigma}\Big) - \frac{\mu\sqrt{k}}{\kappa}\varphi\Big(\frac{\xi}{\kappa}\Big)\Phi\Big(\frac{\mu k\xi - \kappa^2 x}{\sigma\kappa\sqrt{k}}\Big)}{\varphi\Big(\frac{u}{\sqrt{k}}\Big)\Phi\Big(\frac{\xi - \mu u}{\sigma}\Big) - \frac{\mu\sqrt{k}}{\kappa}\varphi\Big(\frac{\xi}{\kappa}\Big)\Phi\Big(\frac{\mu k\xi - \kappa^2 u}{\sigma\kappa\sqrt{k}}\Big)} \tag{12}$$

$$x \ge \frac{u^2}{2}: \quad P(M > x \,|\, X(-\pi) = -\xi) =$$

$$\frac{\varphi\Big(\frac{\kappa^2 x - k\mu\xi}{\sigma\kappa\sqrt{k}}\Big) + \frac{k\mu\xi}{\sigma\kappa\sqrt{k}}\Phi\Big(-\frac{\kappa^2 x - k\mu\xi}{\sigma\kappa\sqrt{k}}\Big)}{\varphi\Big(\frac{\kappa^2 u - k\mu\xi}{\sigma\kappa\sqrt{k}}\Big) + \frac{k\mu\xi}{\sigma\kappa\sqrt{k}}\Phi\Big(-\frac{\kappa^2 u - k\mu\xi}{\sigma\kappa\sqrt{k}}\Big)} \tag{13}$$

The parameter $k > 0$ is defined below and the parameters μ, σ and $\kappa = \sqrt{k\mu^2 + \sigma^2}$ are given by $\mu = e^{-\alpha\pi}$, $\sigma^2 = 1 - e^{-2\alpha\pi}$.

These conditional distributions are derived by use of Bayes' formula and the Slepian model process representation $X_u(\tau) = [u \cos\tau + (u\alpha + Z\sqrt{1+\alpha^2}) \sin\tau] e^{-\alpha\tau} + R(\tau)$ of the ALO response for $\tau > 0$ given an upcrossing of level u for $\tau = 0$. The first term is the linear regression of $X(\tau)$ on $X(0) = u$, $\dot{X}(0) = Z/\sqrt{1+\alpha^2}$ and $R(\tau)$ is a mean zero non-stationary Gaussian residual process. At an arbitrary point in time τ, the derivative $\dot{X}(\tau)$ is independent of $X(\tau)$ and has normal distribution, but sampled over the upcrossings through level u the derivatives make up a Rayleigh distributed population that after multiplication by $\sqrt{1+\alpha^2}$ becomes standard Rayleigh. Thus by assigning the standard Rayleigh distribution $P(Z > x) = e^{-\frac{1}{2}x^2}$, $x \geq 0$ to Z, the process $X_u(\tau)$ becomes a non-Gaussian process that models the behavior of the process $X(\tau)$ when sampled arround upcrossing points. It is noted that the linear regression part is the response of the ALO when started from the displacement u with velocity $Z/\sqrt{1+\alpha^2}$ without external excitation.

Neglecting $\sqrt{1+\alpha^2}$, the maxima of the regression part are obtained for $\tan\tau \approx Z/(u+\alpha Z)$. The value of the square of the first maximum value of $X_u(\tau) - R(\tau)$ after the u-upcrossing then becomes

$$M^2 \approx [u^2 + (\alpha u + Z)^2] \exp\left[-2\alpha \arctan\left(\frac{Z}{u+\alpha Z}\right)\right] \quad (14)$$

from which it is seen that $M^2 \to u^2 + Z^2$ as $\alpha \to 0$. The conditional distribution functions (12) and (13) are derived on the basis of the approximation $M^2 - u^2 = kZ^2$ where k is a function of α and u obtained by substituting the right side of (14) in $(M^2 - u^2)/Z^2$ and setting Z to its unconditional expectation $\sqrt{\pi/2}$ according to the standard Rayleigh distribution. Simulation of sufficiently accurate realizations of M^2 can then be made by first simulating from the distributions (12) or (13) and then correcting slightly by substituting $Z = \sqrt{(M^2 - u^2)/k}$ into (14).

In this way M is defined as the extreme of the ensemble average of the irregularly "micro"-fluctuating response given an upcrossing of the response through level u and given the upcrossing velocity. Obviously this ensemble average acts as a smoothening operator on the response in the vicinity of the upcrossing point.

Except for the sign, a realization of the first clump of approximate plastic displacements of the EPO is simulated by first generating a corrected outcome m_1 from the conditional distribution function (12) with $u = a(1-r)$ [(7) upper bound]. The first plastic displacement $D_1 = rX_1 = ra\xi_1$ is then obtained from (11) setting $\xi_0 = 0$ and $M = m_1$. For $\xi = a(1-r)(1-\xi_1)$ [(7) upper bound] and $u = 0$ a realization m_{02} of M is next generated from the conditional distribution function (13) (for convenience of writing working solely with positive M). If $m_{02} \leq a(1-r)(1-|\xi_1|)$ [(10) upper bound] then the first clump terminates containing the single plastic displacement D_1.

From here the procedure depends on whether $a < 0$ (tension in columns) or $a > 0$ (compression in columns) due to the different positions of the symmetrized elasticity limit in the two cases. The following description corresponds to the case $a > 0$, see Fig. 1, top.

If $m_{02} > a(1-r)(1-|\xi_1|)$ a new corrected realization m_{12} of M is next generated from the conditional distribution function (13) setting $u = a(1-r)(1-|\xi_1|)$. If $m_{12} \leq a(1-r)(1+\xi_1)$ [(7) lower bound (absolute value)] the next displacement minimum is outside the symmetrized elastic domain but still inside the elastic domain. Thus we have $\xi_2 = \xi_1$.

A new independent realization m_{03} of M is generated from (13) for $\xi = m_{12}$ and $u = 0$. If $m_{03} \leq a(1-r)(1-\xi_2)$ [(7) upper bound] the first clump terminates containing the plastic displacements D_1 and $D_2 = 0$. Otherwise a corrected realization m_{13} of M is generated from (13) setting $u = a(1-r)(1-\xi_2)$. Then ξ_3 is calculated from (11) for $M = m_{13}$, and the plastic displacement increment $D_3 = ra(\xi_3 - \xi_2)$ is obtained. The situation is now as after the generation of the first plastic displacement D_1.

If $a(1-r)(1+\xi_1) < m_{12}$ in the second step, then the displacement minimum is in the plastic domain, and ξ_2 is calculated from (11) (with $\dot{x}/|\dot{x}| = -1$) for $M = m_{12}$ giving $D_2 = ra(\xi_2 - \xi_1)$.

Thereafter a realization $M = m_{03}$ is generated from (13) for $\xi = a(1-r)(1+\xi_2)$ [(7) lower bound (absolute value)] and $u = 0$. If $m_{03} \leq a(1-r)(1-|\xi_2|)$ [(10) upper bound], then the first clump terminates containing the two plastic displacement D_1 and D_2 accumulating to the net plastic displacement ξ_2. Otherwise a corrected realization m_{13} of M is generated from (13) for $\xi = a(1-r)(1+\xi_2)$ and $u = a(1-r)(1-|\xi_2|)$. If $a(1-r)(1-|\xi_2|) < m_{13} \leq a(1-r)(1-\xi_2)$ [(7) upper bound] the next displacement maximum is outside the symmetrized elastic domain but still inside the elastic domain giving $\xi_3 = \xi_2$.

A new independent realization m_{04} of M is generated from (13) for $\xi = m_{13}$, $u = 0$. If $m_{04} \leq a(1-r)(1-|\xi_3|)$ [(10) upper bound], then the first clump terminates containing the plastic displacements D_1, D_2 and $D_3 = 0$. Otherwise a new corrected realization m_{14} of M is generated from (13) for $\xi = m_{13}$, $u = a(1-r)(1-|\xi_3|)$, whereupon ξ_4 is calculated from (11) for $M = m_{14}$, and the plastic displacement increment $D_4 = ra(\xi_4 - \xi_3)$ is obtained. The situation is now as after the generation of the first plastic displacement D_1.

3.4 *Distances between clumps*

Based on the level crossing properties of the Cramér-Leadbetter envelope of the ALO response (Ditlevsen & Bognár 1993, Cramér & Leadbetter 1967) it is reported in (Ditlevsen & Bognár 1993) that the distance T_j from the end of the jth clump to the start of the $(j+1)$th clump can be well approximated as π plus a random variable with an exponential distribution of expectation (suppressing index j)

$$\pi + \frac{e^{\frac{1}{2}u^2} - 1}{u\,\delta\,Q(u,\delta)}\sqrt{\frac{2\pi}{1+\alpha^2}} \tag{15}$$

$$Q(u,\delta) \approx 1 -$$

$$2u\int_0^1 \varphi(ux)\Big[1 - \sqrt{2\pi}\frac{2\Phi[\gamma u(1-x^2)] - 1}{2\gamma u(1-x^2)}\Big]dx \tag{16}$$

$$\gamma = \frac{\pi\delta}{2\sqrt{1-\delta^2}} \tag{17}$$

$$\delta^2 = 1 - \frac{1}{1-\zeta_0^2}\Big(1 - \frac{1}{\pi}\arctan\frac{2\zeta_0\sqrt{1-\zeta_0^2}}{1-2\zeta_0^2}\Big)^2 \tag{18}$$

where $\zeta_0 = \zeta(\lambda_0)$. The function $Q(u,\delta)$ is the long run fraction of qualified envelope excursions outside the interval $[-u,u]$, that is, envelope excursions for which there is at least one outcrossing of the ALO response (Ditlevsen 1994) during the excursion. The number $\delta > 0$ is a spectral width parameter of the ALO response,(Vanmarcke 1983) p. 180.

The exponential distribution approximation is less accurate for low crossing levels. Due to the oscillatory nature of the response there is a tendency of having concentrated probability densities around time points separated by π (for symmetric double barrier) in particular for small τ. The corresponding step like nature of the distribution function fades out and approaches an exponential upper tail for increasing τ.

3.5 *Initial conditions*

The initial condition of the EPO compatible to the Slepian modeling of the distribution of the first plastic displacement in a clump is defined by a modification of the distribution of the stationary response of the ALO at an arbitrary point in time, e.g. at $t = 0$. If the displacement of the ALO at time $t = 0$ is outside the elasticity domain, the initial elastic displacement component $X_i(0)$ of the EPO is put equal to the relevant elasticity limit. If in addition the velocity component $\dot{X}_i(0)$ is directed away from the equilibrium position, the EPO is given an instantaneous plastic displacement at $t = 0$ corresponding to setting the plastic work equal to the excess elastic energy of the ALO. Otherwise, if the ALO displacement is inside the elasticity limits, the initial elastic displacement of the EPO is the same as that for the ALO.

The EPO may have any initial plastic displacement that may be included in the definition of the EPO and the corresponding ALO. Thus it is sufficient to assume that the initial plastic displacement is zero.

This initial condition is used to start a direct numerical integration simulation of the EPO response, e.g. as a vectorial Markov sequence with suitably small time steps. The corresponding initial condition for the Slepian simulation is obtained by considering that the stationary ALO response alternates between clumps of excursions outside the elasticity domain of the EPO and separating intervals with oscillations within the elasticity domain. If the time origin is chosen at random on the time axis, then the probability p that the origin falls within a clump of excursions is the ratio of the mean clump duration to the mean duration from the start of a clump to the start of the following clump. A simple approximation is obtained by setting $p = Q(u,\delta)\exp(-u^2/2)$ which is the probability that the Cramér-Leadbetter envelope of the ALO response has a qualified excursion outside the normalized interval $[-u,u]$ at any given point in time. Given that the origin falls within a clump, the first plastic displacement after time zero is simulated by use of (14) and the standard Rayleigh distribution for Z, and is allocated as a jump of the plastic displacement to a time point chosen at random in the interval $]0,\pi/2]$. The sign of the jump is chosen at random. If the origin falls within a separating interval of elastic displacements, the duration to the occurrence of the first clump of plastic displacements can as an approximation be assumed to be exponentially distributed with the expectation (15). Each response starting in a clump is assigned the weight p. Otherwise the weight $1-p$ is assigned to the response.

3.6 *Slepian simulation*

The sample functions of the accumulated plastic displacement process are from the start of the first clump to the start of the second clump approximated by the constant value D_1 over the time interval $T_{01}+]0,\pi]$ (using obvious notation), D_1+D_2 over the time interval $T_{01}+]\pi,2\pi]$, ..., $D_1 + D_2 +$

$\ldots + D_N$ over the time interval $T_{01}+](N-1)\pi, N\pi]$ where N is the clump size. Termination of the clump is then at time $T_{01} + (N-1)\pi$. From time $T_{01} + (N-1)\pi$ to the start of the second clump at time $T_{01} + (N-1)\pi + T_{12}$ the accumulated plastic displacement is $D_1 + D_2 + \ldots + D_N$. The duration T_{12} between the end of the first clump to the start of the second clump is generated by continued simulation of the consecutive alternating troughs/crests and crests/troughs of the ALO response during a suitable number $n-1$ of extra half periods π (e.g. $n = 4$). For each half period π this is done by conditioning both on the crossing of the level $u = 0$ and the simulated trough/crest displacement ξ at $\pi/2$ before the crossing of $u = 0$, that is, by generating a corrected value of M according to (13). If a crossing out of the symmetrized elasticity level $u = (1-r)a(1-r^{-1}|D_1 + D_2 + \ldots + D_N|/|a|)$ occurs within these extra $n-1$ half-periods, then a new clump starts with the first plastic displacement determined by generating a corrected realization of M from (13) for $u = (1-r)a(1-r^{-1}|D_1+D_2+\ldots+D_N|/|a|)$ and ξ equal to the displacement obtained half a period before. If no outcrossing occurs before time $(n+1)\pi$ after the termination of the clump, then a realization of the exponentially distributed random variable T with the expectation (15) corresponding to the level $u = (1-r)a(1-r^{-1}|D_1 + D_2 + \ldots + D_N|/|a|)$ is generated and added to $(n+1)\pi$ to give T_{12}. The sign of the next outcrossing level is then chosen to $\pm(-1)^{[n+1+1/2+T/\pi]}$, where $[x]$ means integer part of x, and +/- is used if the last plastic displacement in the previous clump is positive/negative. This sign rule interpolates between correct choice for small time distances in the mean (low levels) to almost random choice for large time distances in the mean (high levels).

Except for the change of the initial condition for the accumulated plastic displacement from being zero to be $D_1 + D_2 + \ldots + D_N$, the sample function part from the start of the second clump to the start of the third clump and all the following parts are approximated in a completely analogous way. The initial conditions defined in Section 3.5 are taken into account by letting the duration T_{01} be zero with probability p and have the same distribution as T_{12} with probability $1-p$.

In this way a large number of approximate sample functions may be simulated and the development of the distributional behavior can be investigated by statistical analysis of the obtained samples. In the column compression case the sample functions terminate at a last clump where the accumulated plastic displacement passes outside the interval $[-a, a]$. The physical realism is lost long before that time, of course.

For the damping parameter value $\alpha = 0.05$ and the yield level parameter values $A = 1$ and $A = 2$, the thin curves in Fig. 2 (left) are the estimated mean time to first passage of a given plastic displacement level as function of this level for the case of column compression corresponding to the levels $\lambda_0 = 0.4$ and $\lambda_0 = 0.8$. Moreover Fig. 2 (right) shows the estimates of the coefficient of variation, the skewness, and the kurtosis of the first passage time. The results are in the same figure compared with results (thick curves) obtained by direct simulation as described in Section 4.

The three relative statistics, i.e. coefficient of variation, skewness, and kurtosis, show convincingly small deviations between the two sets of results. Also it is seen that the relative statistics are insensitive to the compression force λ and asymptotically for increasing plastic displacement level $|x_p|$ also to the yield level A. For the plastic displacement level $|x_p|$ within a range from 1 to 2, the values of these relative statistics are $V \approx 0.9$, $\alpha_3 \approx 1.6$, and $\alpha_4 \approx 7$. These values are close to those of a random variable of the form T^ν, where T has an exponential distribution and $\nu = 1.15$, that is, a distribution of Weibull type. The Slepian model method is seen to underestimate the mean first passage time by only some few oscillation periods.

4 DIRECT SIMULATION

The white noise force process $W(\tau)$ in (1) and (3) is approximately represented as a sequence of independent Gaussian force impulses $W_1(h), \ldots\ W_n(h), \ldots$ separated in time by h. The nth impulse generates the velocity jump $\dot{X}(nh+) - \dot{X}(nh-) = W_n(h)$ and since the white noise $W(\tau)$ of intensity $I = 4\alpha(1+\alpha^2)$ as $h \to 0$ generates a velocity increment with variance Ih, it follows that we can take $W_n(h) = U_n\sqrt{Ih}$ with $U_1, \ldots U_n, \ldots$ being a sequence of independent standard Gaussian variables.

The vertical ground acceleration entering the equations through the random parameter λ is approximately represented by a stationary sequence of dependent Gaussian random variables $V_1(h), \ldots$ $V_n(h), \ldots$ in the form of a square wave process that between any two consecutive impulses $W_n(h)$ and $W_{n+1}(h)$ is equal to $V_n(h)$. The sequence $V_1(h)$, $\ldots V_n(h), \ldots$ is generated as a second order autoregressive sequence defined later in this section.

The response of the oscillator to this loading sequence is at the time points $\tau = h, 2h, \ldots, nh, \ldots$ given in exact form by the autoregressive vector sequence defined recursively by

$$\mathbf{S}(n+1) = \mathbf{K}(n)\mathbf{S}(n) + \mathbf{Z}(n) \tag{19}$$

where

$$\mathbf{K}(n) = e^{-\alpha h} \begin{bmatrix} \cos \beta_n h & \sin \beta_n h \\ -\sin \beta_n h & \cos \beta_n h \end{bmatrix} \tag{20}$$

$$\mathbf{S}(n) = \begin{bmatrix} \beta_n Y(nh) \\ \dot{X}(nh+) + \alpha Y(nh) \end{bmatrix} \tag{21}$$

$$Y(nh) = \begin{cases} X(nh) - X_{i-1} \\ \quad \text{for (1) (elastic domain)} \\ X(nh) + \text{sign}[\dot{X}(nh+)] \dfrac{12\,A}{\kappa_n|\kappa_n|} \\ \quad \text{for (3) (plastic domain)} \end{cases} \tag{22}$$

$$\beta_n^2 = \begin{cases} \dfrac{g(\kappa_n)}{g(\lambda_0)}(1+\alpha^2) - \alpha^2 \\ \quad \text{for (1) (elastic domain)} \\ \dfrac{\kappa_n|\kappa_n|}{12\,g(\lambda_0)}(1+\alpha^2) - \alpha^2 \\ \quad \text{for(3) (plastic domain)} \end{cases} \tag{23}$$

$$\kappa_n = \lambda_0 \ \text{sign}\Big(1 + \frac{V_n}{\nu - 1}\Big) \sqrt{\Big|1 + \frac{V_n}{\nu - 1}\Big|} \tag{24}$$

$$\mathbf{Z}(n) = \begin{bmatrix} 0 \\ W_n(h) \end{bmatrix} \tag{25}$$

where $\nu\, mg_a > mg_a - P_{cr}$ is an extra static load, see Appendix 2. The first time the event $V_n < (1-\nu)\Big(1 + \dfrac{\pi^2}{\lambda_0|\lambda_0|}\Big)$ occurs, a stability failure results, and the sample function terminates.

It is noted that β_n^2 takes both positive and negative values. Whenever $\beta_n^2 < 0$ we have that $\beta_n = i\sqrt{-\beta_n^2}$ is imaginary, and the complex matrices $\mathbf{K}(n)$ and $\mathbf{S}(n)$ can be replaced by the real matrices

$$\mathbf{K}(n) = e^{-\alpha h} \begin{bmatrix} \cosh |\beta_n| h & \sinh |\beta_n| h \\ \sinh |\beta_n| h & \cosh |\beta_n| h \end{bmatrix} \tag{26}$$

$$\mathbf{S}(n) = \begin{bmatrix} |\beta_n| Y(nh) \\ \dot{X}(nh+) + \alpha Y(nh) \end{bmatrix} \tag{27}$$

where $|\beta_n| = \sqrt{-\beta_n^2}$. For the case $\lambda = 0$ (ideal plasticity corresponding to zero axial force in the columns) the transition matrix $\mathbf{K}(n)$ and the state matrix $\mathbf{S}(n)$ become

$$\mathbf{K}(n) = \begin{bmatrix} 1 & (1 - e^{-2\alpha h})/2\alpha \\ 0 & e^{-2\alpha h} \end{bmatrix} \tag{28}$$

$$\mathbf{S}(n) = \begin{bmatrix} X(nh) + \text{sign}[\dot{X}(nh+)]\, nh\, A/2\alpha \\ \dot{X}(nh+) + \text{sign}[\dot{X}(nh+)]\, A/2\alpha \end{bmatrix} \tag{29}$$

The transition from the elastic state to the plastic state is approximated by adopting the rule that the transition takes place the first time the displacement satisfies the inequality in (4), and the system is then required to satisfy the relevant of the differential equations in (3) as long as the velocity has the same sign. Transition back to the elastic domain is assumed to take place directly after the change of sign of the velocity, and the corresponding plastic displacement is calculated from replacing the inequality sign in (4) by the equality sign.

The sequence of vertical accelerations $V_1(h)$, $\dots V_n(h), \dots$ is herein taken as a stationary Gaussian sequence generated by (19) replacing X and $\dot{X}$ by V and $\dot{V}$, respectively, and using suitable constant values α_0 and β_0 of α and β, respectively. In place of the sequence $W_1, \dots W_n, \dots$ we apply a Gaussian impulse sequence that may be independent of $W_1, \dots W_n, \dots$ or may have any degree of correlation with this sequence.

For the same parameter values as in Section 3.6 sample functions are generated for the vertical random process parameters $\alpha_0 = 0.2$ and $\beta_0 = 1.0$ with a Gaussian excitation sequence independent of the sequence of horizontal excitations. The standard deviation of V is $\sqrt{J/[4\alpha_0(\alpha_0^2 + \beta_0^2)]}$ where $\sqrt{Jh}$ is the standard deviation of the excitation sequence.

Figure 3 shows the mean first passage time for the listed values of J. The curves for $J = 0$ are the same as those shown in Fig. 2. The relative statistics are not presented because they turn out to be insensitive to J, i.e. they are practically the same as those shown in Fig. 2. It is seen that the mean first passage time is shortened by the applied parametric excitation, that is, the vertical excitation has a destabilizing effect in the considered cases. Example calculations not presented herein show that by suitable tuning of the parameter ν of extra static load it is possible to obtain a good approximation without taking the vertical excitation into account, that is, by setting $J = 0$.

5 CONCLUSIONS

The investigations reported herein show that the fast Slepian model process method of simulating the plastic displacement response gives surprisingly accurate information on relevant response statistics for Gaussian white noise excited one degree of freedom elasto-plastic oscillators with softening as well as hardening behavior (results for the hardning case are given in (Ditlevsen & Tarp-Johansen)). A key element of the method is the use of a concept of symmetrized elastic domain.

The actual example is a portal frame with line-

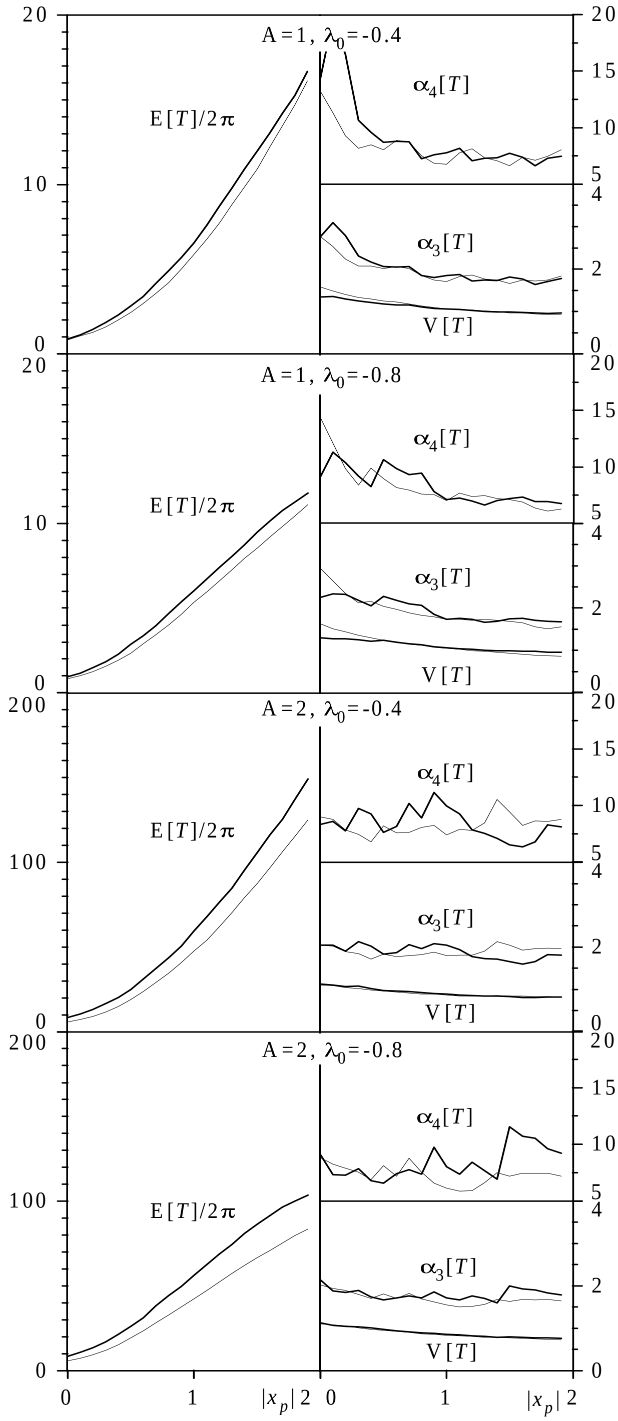

Figure 2. Statistics for first passage time of plastic displacement level under constant column force. Thin curves: Slepian simulation. Thick curves: direct simulation.

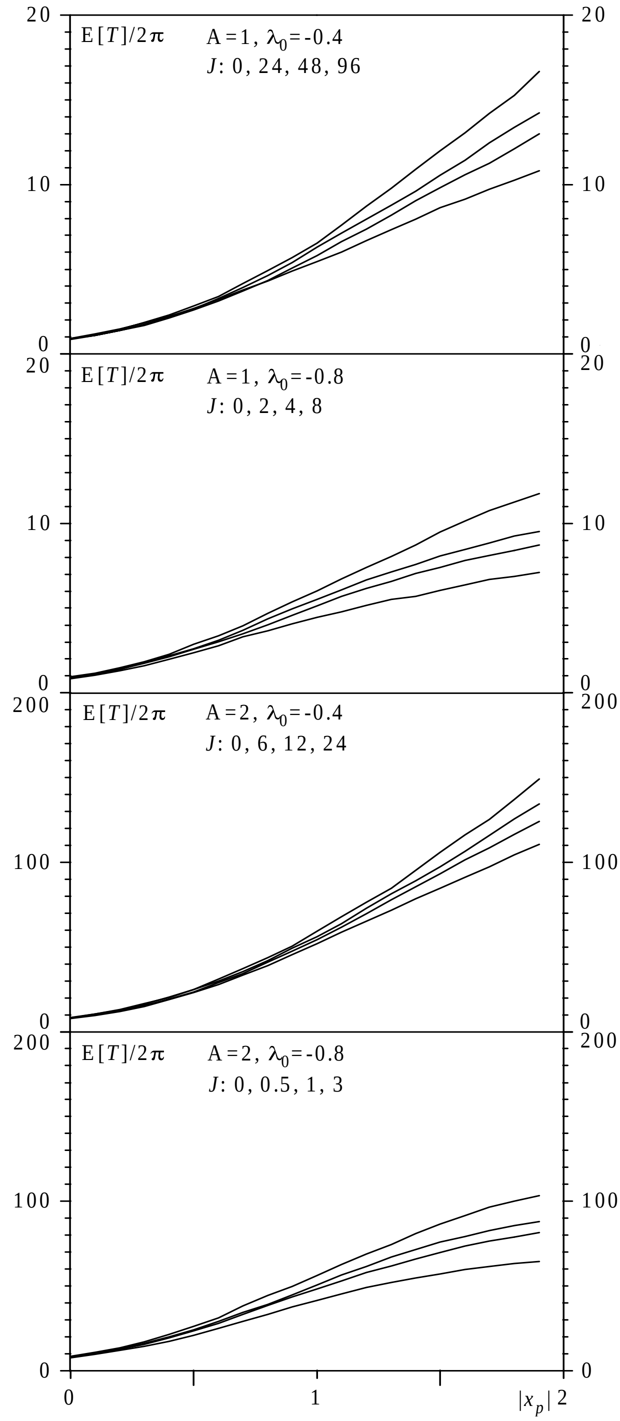

Figure 3. Mean value of first passage time of plastic displacement level under stochastic column force excitations defined by the intensity J.

ar elastic-ideal plastic constitutive behavior of the developing yield hinges in the columns and with a heavy rigid traverse such that the softening behavior is the geometric non-linearity effect of the vertical gravity load. By applying a sufficiently large lifting force on the frame (equivalent to a case of a hanging heavy traverse) the case of geometric hardening behavior is obtained.

The computation time gain factor depends on the choice of parameters and on the choice of time step in the direct simulation. In the present computations the gain factor ranges from 30 to 250 for time steps h between one hundredth and one thousandth of a period of the oscillator. In accordance with the nonlinear dependence of the mean interclump duration on the yield level the gain factor will increase rapidly with increasing u.

The considered example cases with compression in the columns show the interesting result that the relative statistics (coefficient of variation, skewness, kurtosis) of the first passage time of any given reasonable level of plastic displacement are insensitive to change of the axial column force and to the yield level, as well as to the plastic displacement level if this is larger than about 1 (the standard deviation of the ALO). The same is the case when besides the horizontal Gaussian white noise excitation the traverse frame is excited vertically by a random oscillatory excitation defined as a second order filtered Gaussian white noise. Solely the mean first passage time turns out to vary significantly with the mentioned parameter values. Also it turns out that there is a static axial column load, independent of the given plastic displacement level, which is almost equivalent to the vertical excitation in the sense that under this axial static load or under the vertical excitation approximately the same mean first passage time is obtained for the same horizontal white noise excitation.

The example calculations give statistics for the first passage time close to those of a Weibull distribution.

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APPENDIX 1: SHEAR FRAME COLUMN

The shear frame column shown in Fig. 4 is a simple linear-elastic Euler-Bernoulli beam of span L, clamped in at both ends except that the top end is movable horizontally. At both supports there are permanent angular rotations of equal size θ. A vertical load P, positive for tension, negative for compression in the columns, and a horizontal load F directed to the right are applied to the top end of the column. The coordinate along the column axis is x with $x = 0$ at the bottom end and $x = L$ at the top end. The displacement at x is $y(x)$, positive to the right. The bending stiffness is constant along the axis and equal to EI and only the first order beam theory of small displacements is used. Then the differential equation of the displacement $y(x)$ is directly formulated as

$$EI\frac{d^2y(x)}{dx^2}=(L-x)F-M(L)-P[y(L)-y(x)] \qquad (30)$$

The complete solution is

$$y(x) = y(L) + \frac{M(L) - (L-x)F}{P} + C_1 \,\mathrm{co}\left(\lambda\frac{x}{L}\right) + C_2 \,\mathrm{si}\left(\lambda\frac{x}{L}\right) \qquad (31)$$

in which $M(L)$ is the bending moment in the column for $x = L$, and

$$\lambda = \pi\,\mathrm{sign}(P)\sqrt{\frac{|P|}{P_{cr}}},\; P_{cr} = \frac{\pi^2 EI}{L^2} \qquad (32)$$

The functions $\mathrm{co}\,x$ and $\mathrm{si}\,x$ are $\cosh x$ and $\sinh x$, respectively, for $P > 0$, and $\cos x$ and $\sin x$, respectively, for $P < 0$.

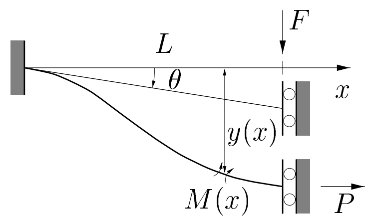

Figure 4. Definition sketch for column

Setting $x = L$ in (31) gives $M(L)$, and setting the derivative $y'(x)$ to θ for $x = 0$ and $x = L$, respectively, gives the values of the arbitrary constants C_1 and C_2. Thus the following results are obtained:

$$\frac{\pi^2 M(L)}{P_{cr}L} = \frac{1-\mathrm{co}\,\lambda}{|\lambda|\,\mathrm{si}\,\lambda}\left(\theta\lambda\,|\lambda| - \frac{\pi^2 F}{P_{cr}}\right) \qquad (33)$$

$$\frac{y(L)}{L} = \theta - \frac{1}{\lambda\,|\lambda|}\left(2\frac{1-\mathrm{co}\,\lambda}{|\lambda|\,\mathrm{si}\,\lambda}+1\right)\left(\theta\lambda\,|\lambda| - \frac{\pi^2 F}{P_{cr}}\right) \qquad (34)$$

The right sides of (33) and (34) approach $\pi^2 F/2P_{cr}$ and $\theta+\pi^2 F/12\,P_{cr}$, respectively, as $\lambda \to 0$. Solving (34) with respect to $\pi^2 F/P_{cr}$ gives

$$\frac{\pi^2 F}{P_{cr}} = 12\,g(\lambda)\Big(\frac{y(L)}{L} - \theta\Big) + \theta\lambda\,|\lambda| \tag{35}$$

$$g(\lambda) = \frac{\lambda^3\,\mathrm{si}\,\lambda}{12\,(2{-}2\,\mathrm{co}\,\lambda{+}|\lambda|\,\mathrm{si}\,\lambda)} \sim 1{+}\frac{1}{10}\lambda\,|\lambda| \tag{36}$$

as $\lambda \to 0$. Yield hinge formation starts at the two supports of the column whenever the bending moment $M(L)$ reaches either M_y or $-M_y$ where M_y is the yield bending strength of the beam, assumed to be the same independent of the normal force P in the column. The corresponding two yield limit horizontal forces F are obtained from (33). Upon substitution in (34) the elasticity displacement limits are obtained as

$$\theta L - h(\lambda)\frac{\pi^2 M_y}{6\,P_{cr}} \le y(L) \le \theta L + h(\lambda)\frac{\pi^2 M_y}{6\,P_{cr}} \tag{37}$$

$$h(\lambda) = \Big(g(\lambda){-}\frac{1}{12}\lambda\,|\lambda|\Big)^{-1} \sim 1{-}\frac{1}{60}\lambda\,|\lambda| \tag{38}$$

as $\lambda \to 0$. In the plastic range the force F is obtained by a simple moment equilibrium condition as

$$\frac{\pi^2 F}{P_{cr}} = \lambda\,|\lambda|\frac{y(L)}{L} + \mathrm{sign}[\dot y(L)]\frac{2\pi^2 M_y}{P_{cr} L} \tag{39}$$

for $\mathrm{sign}[\dot y(L)]y(L) > \mathrm{sign}[\dot y(L)]\theta L + h(\lambda)\dfrac{\pi^2 M_y}{6\,P_{cr}}$.

APPENDIX 2: EQUATION OF MOTION

Let the traverse of a single degree of freedom shear frame be infinitely stiff and have mass m, and let the traverse be carried by two copies of the column described in Appendix 1. Yield hinge formation develops in the columns at the four supports whenever the common bending moment reaches either M_y or $-M_y$ where M_y is the yield bending strength, assumed to be the same independent of the normal force P in the columns. Let the traverse be dynamically excited by white noise ground acceleration $g_a\ddot U, g_a\ddot V$, horizontally and vertically, respectively, at both supports, where g_a is the acceleration of gravity. Then the equation of motion of the traverse mass becomes

$$g_a\ddot U(t) + \ddot x(t) + 2\zeta(\lambda)\omega_0(\lambda)\dot x(t) + \frac{F(t)}{m} = 0 \tag{40}$$

where the force $F(t)$ is given by (35) and (39) with $F(t)$ and $x(t)$ substituted for F and $y(L)$, respectively, $P = mg_a(\nu{-}1{+}\ddot V)$ where $\nu m g_a$ is a possible extra static force on top of the gravity force $-mg_a$ from the traverse, and

$$\omega_0(\lambda) = \sqrt{\frac{12 P_{cr}}{\pi^2 m L}}\sqrt{g(\lambda)}, \quad \zeta(\lambda) = \frac{\zeta}{\sqrt{g(\lambda)}} \tag{41}$$

where $\zeta = \zeta(0)$ is the viscous damping ratio, and $\omega_0(0)$ is the angular eigenfrequency of the undamped linear oscillator obtained by setting $\zeta = 0$, $\lambda = 0$ and letting $M_y \to \infty$. In the elastic range the equation of motion can thus be written as

$$\ddot x + 2\zeta(\lambda)\omega_0(\lambda)\dot x + \omega_0^2(\lambda)x = -g_a\ddot U + \theta L\,\omega_0^2(\lambda)\Big(1 - \frac{\lambda\,|\lambda|}{12\,g(\lambda)}\Big) \tag{42}$$

In the plastic range the equation of motion becomes, Appendix 1,

$$\ddot x + 2\zeta(\lambda)\omega_0(\lambda)\dot x + \frac{\lambda\,|\lambda|}{12}\omega_0^2(0)x = -g_a\ddot U - \mathrm{sign}(\dot x)\frac{2M_y}{mL} \tag{43}$$

for $\mathrm{sign}(\dot x)\,x > \mathrm{sign}(\dot x)\,\theta L + h(\lambda)\dfrac{\pi^2 M_y}{6\,P_{cr}}$. Besides being externally excited by the force $-mg_a\ddot U$, it is seen that the oscillator is parametrically excited through the function $\lambda(t)$.

The normalized and dimensionless representation of the response considered herein is obtained from the response of the associated linear oscillator (ALO) corresponding to $\theta = 0$ and $\lambda(t) = \lambda_0$ (constant). In the dimensionless time $\tau = \omega_0(\lambda_0)\sqrt{1-\zeta^2(\lambda_0)}\,t$ this ALO has the period 2π. The dimensionless response is defined as $X(\tau) = x(t)/\sigma$, where $\sigma^2 = \pi S/4\zeta(\lambda_0)\,\omega_0^3(\lambda_0)\,m^2$ is the stationary variance of the response of the ALO under the assumption that the horizontal excitation $-mg_a\ddot U(t)$ is stationary Gaussian white noise of zero mean and intensity πS, (Ditlevsen & Bognár 1993). Applying these variable transformations to the differential equation for $x(t)$ gives the differential equation for $X(\tau)$ as written out in Section 2 with

$$A = 4g(\lambda_0)\frac{M_y}{L}\sqrt{\frac{\zeta(\lambda_0)}{\pi S\omega_0(\lambda_0)}} \tag{44}$$