EXTREME OF RANDOM FIELD OVER RECTANGLE WITH APPLICATION TO CONCRETE RUPTURE STRESSES

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Abstract

Probabilities of excursions of random processes and fields into critical domains are of fundamental interest in many civil engineering decision problems. Examples are reliability evaluations of structures subject to random load processes, the influence of the size of a structural element on the carrying capacity of the element when the local material properties vary as a random field over the element, evaluation of the risk of floods or draughts in connection with river and reservoir management, etc. Only for some very few special cases of process and field models it is possible to obtain exact results for such probabilities. However, due to the engineering importance of the problem, several approximate assessment methods have been suggested in the past. The suitability and accuracy of each of these methods depends on the type of process or field under consideration. Often recourse must be taken to time consuming simulation procedures. This paper revives a conceptually simple approach that gives surprisingly good results in particular for wide band types of random processes and fields. The principle of the method is published in the papers (**?**, **?**) together with applications to various simple scalar process examples. However, the potential generalization properties of the method have never been properly pointed out except in a dissertation of limited circulation (**?**). The generalization concerns vector processes and fields. The derivation made in this paper is simpler and more elegant than that used in the dissertation. The closed form formulas obtained for smooth Gaussian fields over rectangles contain size effects both with respect to the area of the rectangle and the side lengths of the rectangle. Published rupture stress data for plain concrete beams illustrate the applicability of the derived closed form extreme value distributions as models for distributions of rupture stresses related to weakest link mechanisms.

Probability approximation for extreme values

Let $X(t)$ be a random process and let $u(t)$ be a non-random function. The probability $\psi(s,t) = P\{\forall\tau \in [t, t+s\,] : X(\tau) < u(\tau)\}$ is the object of evaluation based on approximate reasoning. Assume that $\psi(s,t)$ is never zero. Then the function $c(r,s,t)$ defined by

$$\psi(r+s, t-r) = c(r,s,t)\psi(r,t-r)\psi(s,t) \tag{1}$$

is a measure of the dependence between the events $\{\forall\tau \in [t-r, t\,] : X(\tau) < u(\tau)\}$ and $\{\forall\tau \in [t, t+s\,] : X(\tau) < u(\tau)\}$. It is seen that $c(r,0,t) = c(0,s,t) = 1/\psi(0,t)$. By a simple change of variables in (1) we get the equation

$$\psi(s,t) = c(r, s-r, t+r)\psi(r,t)\psi(s-r, t+r) \tag{2}$$

and thus

$$\psi_{,1}(s,t) = c(r, s{-}r, t{+}r)\psi(r,t)\psi_{,1}(s{-}r, t{+}r) + c_{,2}(r, s{-}r, t{+}r)\psi(r,t)\psi(s{-}r, t{+}r) \tag{3}$$

from which the equation

$$\frac{\psi_{,1}(s,t)}{\psi(s,t)} = \frac{\psi_{,1}(0,t+s)}{\psi(0,t+s)} + \psi(0,t+s)c_{,2}(s,0,t+s) \tag{4}$$

is obtained by the limit passage $r \to s$. This equation shows that $c_{,2}(0,0,t) = 0$. Integration gives

$$\psi(s,t) = \psi(0,t)\exp\Big(\int_t^{t+s} \frac{\psi_{,1}(0,\tau)}{\psi(0,\tau)}\, d\tau\Big)\, H(s,t) \tag{5}$$

where $H(s,t) = \exp\Big(\int_0^s \psi(0,t+s)c_{,2}(s,0,t+s)\, ds\Big)$ has the properties $H(0,t) = 1$ and $H_{,1}(0,t) = 0$. Thus $H(s,t)$ may be considered as a correction factor. Simulations show that this correction factor is often of minor importance. In the following the approximation $H(s,t) \equiv 1$ will be used. Since

$$\psi(s,t) = P[X(t) < u(t), X(t+s) < u(t+s)] + o(s), \quad \lim_{s \downarrow 0} \frac{o(s)}{s} = 0 \tag{6}$$

under general regularity conditions, it is seen that

$$\psi_{,1}(0,t) = \frac{\partial}{\partial s} P[X(t) < u(t), X(t+s) < u(t+s)]_{s=0} =$$
$$\frac{\partial}{\partial s}\{P[X(t) < u(t)] - P[X(t) < u(t), X(t+s) \geq u(t+s)]\}_{s=0} = -\nu(t) \tag{7}$$

where $\nu(t)$ is the upcrossing rate of the process X through the level u at time t. By substitution into (5) we thus get the result

$$P\{\forall\tau{\in}[t, t{+}s\,]{:}X(\tau){<}u(\tau)\} \approx P[X(t){<}u(t)]\exp\Big(-\int_t^{t+s} \frac{\nu(\tau)}{P[X(\tau){<}u(\tau)]}\, d\tau\Big) \tag{8}$$

For high levels (i.e. $\forall t : P[X(t) < u(t)] \approx 1$) this result corresponds to the result obtained by assuming the the stream of upcrossings is an inhomogeneous Poisson process with intensity $\nu(t)$.

The following consideration generalizes the result (8) to the problem of assessing the probability $P\{\forall\tau \in [t, t+s\,] : X_1(\tau) < u_1(\tau), \ldots, X_n(\tau) < u_n(\tau)\}$ where $X_1(t), \ldots, X_n(t)$ are n possibly jointly dependent processes, and $u_1(t), \ldots, u_n(t)$ are non-random functions. Let the n processes together possess the property that for any t the probability is $o(h)$ to get the event $\{X_i(t_i) = u_i(t_i)\} \cap \{X_j(t_j) = u_j(t_j)\}$ where $t < t_i, t_j < t + h$. Given that a crossing of process X_i through level u_i occurs at time t, then this crossing is relevant for the problem if and only if $X_j(t) < u_j(t)$ for all $j \neq i$. Thus the relevant crossing rate contribution from process X_i at time t is $\nu_i(t)\, P[\forall j \neq i : X_j(t) < u_j(t) \,|\, X_i(t) = u_i(t)]$ where $\nu_i(t)$ is the unthinned crossing rate of X_i through level u_i. Thus (8) gives

$$
\begin{aligned}
&P[\forall \tau \in [t, t+s\,]\, \forall i \in \{1, \ldots, n\} : X_i(\tau) < u_i(\tau)] \\
&\approx P[X_1(t) < u_1(t), \ldots, X_n(t) < u_n(t)] \\
&\cdot \exp\Big(-\int_t^{t+s} \frac{\sum_{i=1}^n \nu_i(\tau)\, P[\forall j \neq i : X_j(t) < u_j(t) \,|\, X_i(t) = u_i(t)]}{P\{X_1(\tau) < u_1(\tau), \ldots, X_n(\tau) < u_n(\tau)\}}\, d\tau\Big)
\end{aligned} \tag{9}
$$

A step further is the problem of assessing the probability $P\{\forall(\tau_1, \tau_2) \in [t_1, t_1 + s_1\,] \times [t_2, t_2 + s_2\,] : X(\tau_1, \tau_2) < u(\tau_1, \tau_2)\}$ where $X(t_1, t_2)$ is a random field over $\mathbb{R}^2$ and $u(\tau_1, \tau_2)$ is a suitably regular function of two variables. According to the principle of (5) with $h \equiv 0$ we have

$$
\begin{aligned}
&P\{\forall(\tau_1, \tau_2) \in [t_1, t_1 + s_1\,] \times [t_2, t_2 + s_2\,] : X(\tau_1, \tau_2) < u(\tau_1, \tau_2)\} \\
&\approx P\{\forall \tau_2 \in [t_2, t_2 + s_2\,] : X(t_1, \tau_2) < u(t_1, \tau_2)\} \\
&\exp\Big(\int_{t_1}^{t_1+s_1} \lim_{h \downarrow 0} \frac{P\{\forall \tau_2 {\in} [t_2, t_2{+}s_2\,] {:} X(\tau_1, \tau_2) {<} u(\tau_1, \tau_2), X(\tau_1{+}h, \tau_2) {<} u(\tau_1{+}h, \tau_2)\}}{h\, P\{\forall \tau_2 {\in} [t_2, t_2{+}s_2\,] {:} X(\tau_1, \tau_2) {<} u(\tau_1, \tau_2)\}}\, d\tau_1\Big)
\end{aligned} \tag{10}
$$

The following results are obtained by use of the previous formulas: Assume that $X(s, t)$ is a homogeneous Gaussian random field of mean 0, variance 1, and twice differentiable correlation function Cov$[X(0, 0), X(s, t)] = \rho(s, t)$. Then for constant $u(s, t) = u$:

$$
\begin{aligned}
&P\{\forall(\tau_1, \tau_2) \in [0, s\,] \times [0, t\,] : X(\tau_1, \tau_2) < u\} \\
&\approx \Phi(u) \exp\Big\{-\frac{\varphi(u)}{2\pi\Phi(u)}\Big[(\gamma_1 s + \gamma_2 t)\sqrt{2\pi} + \Big(u + \frac{\varphi(u)}{\Phi(u)}\Big)\gamma_1\gamma_2 st\Big]\Big\}
\end{aligned} \tag{11}
$$

where $\gamma_1 = \sqrt{-\rho_{,11}(0, 0)}$, $\gamma_2 = \sqrt{-\rho_{,22}(0, 0)}$. Assume that $X(s, t)$ is a Gaussian random field of mean 0, variance 1, which is homogeneous in the direction of s and has a twice differentiable correlation function of the form Cov$[X(0, \tau), X(s, \tau + t)] = \rho_1(s)\rho_2(t, \tau)$. Then for $u(s, t) = u(t)$ = constant in the direction of s, one obtains

$$
\begin{aligned}
&P\{\forall(\tau_1, \tau_2) \in [0, s\,] \times [0, t\,] : X(\tau_1, \tau_2) < u(\tau_2)\} \approx \Phi[u(0)] \exp\Big(-\gamma_1 s \frac{\varphi[u(0)]}{\sqrt{2\pi}\Phi[u(0)]}\Big) \cdot \\
&\exp\Big\{\int_0^t \Big[-\gamma_2(\tau)\varphi\Big(\frac{u'(\tau)}{\gamma_2(\tau)}\Big) {+} u'(\tau)\Phi\Big(-\frac{u'(\tau)}{\gamma_2(\tau)}\Big)\Big] \frac{\varphi[u(\tau)]}{\Phi[u(\tau)]}\Big[1 {+} \gamma_1 s\Big(\frac{\varphi[u(\tau)]}{\Phi[u(\tau)]} {+} u(\tau)\Big)\Big] d\tau\Big\}
\end{aligned} \tag{12}
$$

where $\gamma_1 = \sqrt{-\rho_1''(0)}$, $\gamma_2(\tau) = \sqrt{-\rho_{2,11}(0, \tau)}$. The above results have all been compared to simulation estimates of the considered probabilities for wide band types of smooth Gaussian processes and fields (**?**). For such processes the formulas give quite good results.

Application to plain concrete rupture stress data

A series of test results for bending rupture of plain concrete beams of square cross-section and four different widths $b = 6, 9, 12, 18$ inches is published in (**?**) as test series I. The beam span was three times the beam width. Each beam was simply supported and subject to equal loads at the third points, uniformly distributed over the width of the beam. Corresponding to the load at rupture the rupture stress was defined as the maximal tension stress calculated from simple beam theory with linear stress variation over the cross-section. Thus this maximal tension stress is constant within the $b \times b$ square of the beam surface between the two vertical planes of the loads. Assuming that brittle fracture starts out from a point of this square, a statistical size effect on the rupture stress will be observed only if there is a random variation of the tensile strength over the square. Since the test results show a clear size effect it is plausible to assume that the tensile strength in the direction of the beam axis can be modeled as the sum of a random strength, constant over the square, and a homogeneous random field of zero mean. In the following statistical analysis it is assumed that the random field is a suitably smooth Gaussian field. Such smoothness is consistent with the assumption that a small crack only will develop into a fast growing large crack if some local average of a possibly strongly fluctuating micro stress field exceeds a critical local value defined by the smooth Gaussian field.

There is evidence in the data that a random variable should be added to the random field. In fact, each set of four beams of different size were cast as companions from the same concrete mix. This introduces correlation between the rupture stresses for the different sized beams. From the data (test series I) correlation coefficients as large as 0.90 to 0.95 are estimated. This phenomenon can be modeled by a hierarchical model of the form $Z_i = \mu + Y + \sigma \min X_i(s,t)$, where μ, σ are constants, Z_1, Z_2, Z_3, Z_4 are the random rupture stresses for the four beam sizes, respectively, $X_i(s,t)$ are the random fields of zero mean and unit standard deviation all assumed to be isotropic with the same correlation length parameter γ, mutually independent and independent of the random variable Y. The minimum is taken over the square of maximal tension stress. It follows that $\text{Var}[Y] = \text{Cov}[Z_i, Z_j]$ for all $i \neq j$. Estimating from the data gives the standard deviation $D[Y] \approx 69$ psi.

Thus $E[Z_1] - E[Z_2] = [f(3;\gamma) - f(2;\gamma)]\sigma$, $E[Z_1] - E[Z_3] = [f(4;\gamma) - f(2;\gamma)]\sigma$, $E[Z_1] - E[Z_4] = [f(6;\gamma) - f(2;\gamma)]\sigma$, where $f(2;\gamma), f(3;\gamma), f(4;\gamma)$, and $f(6;\gamma)$ are the expectations of the maximum of the Gaussian field over squares of side length $b = 6, 9, 12, 18$ in, respectively. By eliminating σ the two equations

$$\frac{E[Y_1] - E[Y_3]}{E[Y_1] - E[Y_2]} = \frac{f(4;\gamma) - f(2;\gamma)}{f(3;\gamma) - f(2;\gamma)}, \quad \frac{E[Y_1] - E[Y_4]}{E[Y_1] - E[Y_2]} = \frac{f(6;\gamma) - f(2;\gamma)}{f(3;\gamma) - f(2;\gamma)} \tag{13}$$

are obtained. With the left hand sides assigned values that are consistent with the model, these two equations are identical, of course. If the mean values are replaced by the averages of the observations there are no solutions because the left hand sides of both equations turn out to be smaller than the minimal values of the right hand sides. These minimal values are taken for both expressions almost at the same value of γ well approximated by the value $\gamma = 1.00$ in^{-1}. By using a principle of making the smallest possible additive corrections to the averages in the left hand sides (in the sense of suitably weighted Euclidian distance) to satisfy the equations for $\gamma = 1.00$ in^{-1}, corrections of the order of the statistical uncertainty of the average estimates are obtained. In fact, the estimates are corrected from the averages 650, 592, 572, 550 psi to the estimates $E[Z_1] \approx 640$, $E[Z_2] \approx 600$, $E[Z_3] \approx 575$, $E[Z_4] \approx 542$ psi, respectively. Thereafter the estimates of $\sigma \approx 108$ psi and $\mu \approx 874$ psi are readily obtained from the previous equations.

The distribution functions shown in Fig. 1 (left) are obtained for Z_1, Z_2, Z_3, Z_4 by a convolution operation assuming that the X is Gaussian and that $-\min X_i(s,t) = \max X_i(s,t)$ has the distribution function (11) with $\gamma_1 = \gamma_2 = \gamma$ from which the expectations $f(2;\gamma)$, $f(3;\gamma)$, $f(4;\gamma)$,$f(6;\gamma)$ were obtained by integration from 0 to ∞ with respect to u. It is seen that a quite good fit is obtained by the proposed random field model. Interestingly, the test beams were cast from pavement concrete using a medium grade 1-in max sized manufactured aggregate, that is, with a maximal aggregate size approximately equal to the correlation length γ^{-1} obtained from the rupture stress data.

Lindner and Sprague have also made a second test series on the basis of the same concrete mix recipe. This series II solely consists of $b = 6$ and 9 in wide beams in companions of four beams, two of each size and of length $3b$. For each set of companion beams and for each beam width a test with two equal loads at the third points and a test with central loading was made. In the case of third point loading the same distribution of the rupture stress as in test series I should apply. In Fig. 1 (right, top) the empirical distributions for third point loading are plotted together with the corresponding distribution functions from Fig. 1 (left). It is seen that the size effect prediction holds in the mean. However, whatever is the reason, for the 9 in beams the standard deviation of the empirical results is considerably smaller than for the 9 in beams in test series I.

To compare the obtained field model with the test results for central loading the formula (10) is applied with $u(\tau) = (Y - 2x\tau/L)/\sigma$ for $0 \leq \tau \leq L/2$ and $u(\tau) = (Y - 2x(L-\tau)/L)/\sigma$ for $\leq L/2 \leq L$ where $L = 3b$. For $\gamma_2(\tau) = \gamma_1 = \gamma$ this gives the conditional distribution function

$$F_Z(x\,|\,Y) \approx 1 - \Phi(\xi)\Big(\frac{\Phi(\xi)}{\Phi(\eta)}\Big)^A \exp\Big(-\gamma b(1+A)\frac{\varphi(\xi)}{\sqrt{2\pi}\Phi(\xi)}\Big) \tag{14}$$

$$A = \frac{3b\gamma}{\eta - \xi}\varphi\Big(\frac{2(\eta-\xi)}{3b\gamma}\Big) - 2\Phi\Big(-\frac{2(\eta-\xi)}{3b\gamma}\Big), \quad \xi = \frac{Y-x}{\sigma}, \quad \eta = \frac{Y}{\sigma} \tag{15}$$

The comparison with the data is made in Fig. 1 (right, bottom). The full curves correspond to the parameters estimated from test series I. The mean size effect is very well predicted and for the case of the 6 in beams also the distribution function fits fairly well. As for the third point loading case, the standard deviation of the data sample for the 9 in beams is considerably smaller than predicted. The dotted curves are obtained by setting the variance of the random variable part Y of the rupture strength field to zero, keeping the values of all other parameters as before. As it should be expected this has negligible influence on the mean size effect, while the standard deviation is decreased. The estimate of the variance of Y obtained directly from series II is not zero, but somewhat smaller than that of series I.

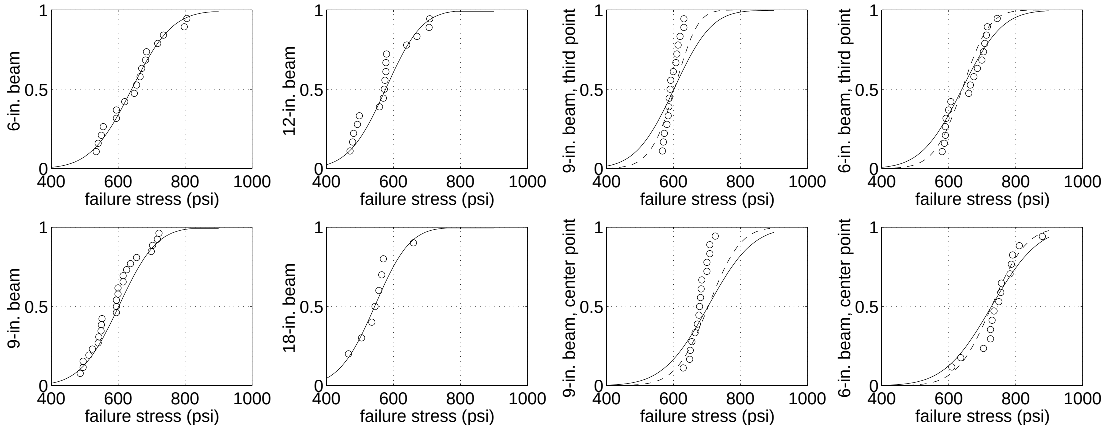

Figure 1: Left four diagrams: Field model distribution functions of rupture stress for third point loading fitted to the shown series I data distributions. Right four diagrams: Distribution functions (full curves) for both third point loading and center point loading for the same parameter values as in the left four diagrams compared to the series II data distributions. The dashed curves are for the variance set to zero of the random variable part of the field model and all other parameter values kept unchanged.

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