Slepian modeling as a computational method in random vibration analysis of hysteretic structures

Ove Ditlevsen, Niels Jacob Tarp-Johansen
Department of Structural Engineering and Materials, Building 118, Technical University of Denmark, DK 2800 Lyngby, Denmark, e-mail: od@bkm.dtu.dk and njtj@bkm.dtu.dk

ABSTRACT: The Slepian model process method combined with envelope level crossing properties proves its usefulness as a tool for fast simulation of approximate plastic displacement responses of a wide class of elasto-plastic oscillators (EPOs) of one or more degrees of freedom excited by stationary Gaussian white noise. The computation time for obtaining estimates of relevant statistics on a given accuracy level is decreased by factors of one or more orders of size as compared to the computation time needed for direct elasto-plastic displacement response simulations by vectorial Markov sequence techniques. Moreover the Slepian method gives valuable physical insight about the details of the plastic displacement development by time. The paper gives a general self-contained mathematical description of the Slepian method based plastic displacement analysis of Gaussian white noise excited EPOs. Experiences with the method are reported with reference to earlier papers. Finally the convincing accuracy of the method is illustrated by an example of one degree of freedom EPOs with hardening or softening plastic behavior.

1 INTRODUCTION

The Slepian model method has in several papers proven worthy to be on the official list of methods apt for solving problems related to non-linear randomly excited oscillators. However, the Slepian model method has still not been generally appreciated in the literature.

Besides a variety of direct simulation methods, the official list (see references in (Ditlevsen & Bognár 1993, Soong & Grigoriu 1993, Lin & Cai 1995)) contains the class of methods by which the actual non-solvable possibly vectorial stochastic differential equation of motion is replaced by a solvable differential equation using some least error principle (equivalent linearization or non-linearization). Another class of methods on the list are about finding approximate solutions to the unclosed infinite hierarchy of moment equations by making the system of equations finite by some closure principle. A third class of methods aims at formulating stochastic differential equations that approximately describe the amplitudes and phases as diffusion processes. These methods are characterized as stochastic averaging methods. All of the methods on the official list basically look for either just the marginal moments up to some order, or they go after the more ambitious goal of calculating some of the finite-dimensional distributions of the stationary or non-stationary response by solving the associated Fokker-Planck equation or by using different versions of the so-called integral path method. For many problems an approximate analysis is practicable only for the stationary response. In all cases a rather low dimensionality of the physical problem sets a limit to the applicability of these methods.

An elasto-plastic oscillator is herein defined to be an oscillator of one or more degrees of freedom that within a neighborhood of the zero response behaves like a linear oscillator and outside this neighborhood has hysteretic behavior. Subject to Gaussian white noise the response will be piecewise conditionally Gaussian. As the white noise intensity approaches zero the response will asymptotically become unconditionally Gaussian. This Gaussianity property makes it possible to apply the strongly simplifying Slepian model process concept. A Slepian model process is the sum of a non-Gaussian linear regression part of random variable form and a Gaussian residual process. It describes the probabilistic structure of a set of conditional responses of the linear oscillator associated to the elasto-plastic oscillator when extending the linearity domain to the entire space. The conditioning events are defined as given response properties at an outcrossing to the plastic domain (such as a given outcrossing velocity) or at one or more successive local response extremes.

Except for the often very time consuming direct simulation methods most of the methods on the official list are not well suited for exploring the tail behavior of the response. From an engineering point of view this is a drawback, of course. For hysteretic oscillators the approximate analytical methods in general give no answer to the problem of separating the process of plastic displacements from the elastic response. What is needed for this problem is essentially to study the behavior of the individual sample functions and not just the probability measure over the set of all sample functions. Time consuming simulation methods generate the individual sample functions and observe the occurrence of the different events of interest, of course. However, the Slepian model method offers an approximate analytical reasoning that concentrate on sample function behavior in the tail regions. Besides being a powerful method that in some cases even may lead to approximate closed form distribution solutions, the method has a direct physical appeal in contrast to the probability measure oriented methods of the official list.

The terminology Slepian model process has been coined by Lindgren in recognition of the works of Kac and Slepian (Kac & Slepian 1959) in which the important horizontal window crossing concept is first introduced. Together with his co-workers, Lindgren has studied several interesting engineering applications and a great variety of other problems involving many types of mathematical generalizations. An introduction to Slepian model process theory in a mathematically stringent setting has been given in (M. R. Leadbetter & Rootzen 1983), which also contains a list of references to papers that deal with applications.

Using the idea of the associate linear oscillator approach first introduced by Karnopp & Scharton 1966 (Karnopp & Scharton 1966) and applied among others in works by Vanmarcke & Veneziano 1974 , Iyengar & Iyengar 1978, Grossmayer & Iwan 1981, Ziegler & Irshick 1985 (references given in (Ditlevsen & Bognár 1993)), the papers (Ditlevsen 1986, Ditlevsen & Lindgren 1988, Ditlevsen 1991, Ditlevsen 1994) report on applications of the Slepian model process method to obtain information about different features concerning probability distributions related to the plastic displacement response of the ideal elasto-plastic oscillator of one degree of freedom excited by Gaussian white noise. The final success for using the Slepian model process method to make a purely theoretical derivation of accurate approximations to the plastic displacement distributions is reported in (Ditlevsen & Bognár 1993). Plastic displacement distributions derived by Roberts (Roberts 1980) by the stochastic averaging method have a completely different mathematical form from those derived in (Ditlevsen & Bognár 1993). Nevertheless, the numerical differences turn out to be surprisingly small.

Attempts to generalize the method to more than one degree of freedom reveal several obstacles of theoretical and computational nature. These difficulties have been analyzed and evaluated with respect to numerical importance for a multi-story shear frame type of ideal elasto plastic oscillator. The example calculations demonstrate that principles almost as simple as for one degree of freedom can be applied leading to accurate plastic displacement distribution assessments (Ditlevsen & Randrup-Thomsen 1996, Randrup-Thomsen & Ditlevsen 1997, Randrup-Thomsen 1997).

The plastic displacement processes of Gaussian white noise excited non-ideal plastic oscillators can also be simulated by the Slepian method. Compared to direct response simulation the Slepian method is very fast and it gives a surprisingly accurate assessments of the plastic displacement distributions as well as of the properties of the plastic displacement process (Randrup-Thomsen & Ditlevsen 1998, Ditlevsen & Tarp-Johansen 1997, Ditlevsen, Tarp-Johansen, & Randrup-Thomsen 1998).

This paper reviews the Slepian model process method and reports on the experiences about the capabilities of the method to obtain information about the plastic response of hysteretic oscillators of one or more degrees of freedom excited by Gaussian white noise. The only necessary assumption for the applicability of the method in its present form is that the equations of motion of the oscillator have a domain of approximate linearity within which the oscillation solutions persist for a time duration of several times the period of the linear oscillations. The paper is self-contained with respect to the necessary regression and random process theory for the reader to appreciate the method including its physical interpretation in relation to the linear white noise excited oscillator.

2 ASSOCIATE LINEAR OSCILLATOR

For a large class of hysteretic oscillators with a finite number of degrees of freedom subject to external white noise forcing $\mathbf{W}(t)$ the random displacement process $\mathbf{X}(t)$ is governed by the differential equation

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{F}(\mathbf{X}_p, \mathbf{X} - \mathbf{X}_p, \dot{\mathbf{X}}) = \mathbf{W} \tag{1}$$

where $\mathbf{M}$ is the mass matrix and $\mathbf{F}(\mathbf{X}_p, \mathbf{X}-\mathbf{X}_p, \dot{\mathbf{X}})$ is the restoring force defined as a function of the plastic displacement part $\mathbf{X}_p(t)$ of the total displacement $\mathbf{X}(t)$, the elastic part $\mathbf{X}(t) - \mathbf{X}_p(t)$, and the velocity $\dot{\mathbf{X}}(t)$.

Let $\mathbf{D}$ and $\mathbf{K}$ be given positive definite matrices and let $\mathcal{E}(\mathbf{0};t)$ be a given open domain that contains the origin $\mathbf{0}$ as an internal point of $\mathcal{E}(\mathbf{0};t)$. In case the restoring force has the linear form $\mathbf{F}(\mathbf{X}_p, \mathbf{X}-\mathbf{X}_p, \dot{\mathbf{X}}) = \mathbf{D}\dot{\mathbf{X}}+\mathbf{K}(\mathbf{X}-\mathbf{X}_p)$ for $\mathbf{X}-\mathbf{X}_p \in \mathcal{E}(\mathbf{0};t)$, $\mathbf{F}$ is hysteretic for $\mathbf{X}-\mathbf{X}_p \in \mathbb{R}^n \setminus \mathcal{E}(\mathbf{0};t)$, and $\mathbf{F}$ is continuous at the boundary $\partial\mathcal{E}(\mathbf{0};t)$, then the oscillator is called an elasto-plastic oscillator (EPO).

The domain $\mathcal{E}(\mathbf{0};t)$ is called the elasticity domain and the boundary $\partial\mathcal{E}(\mathbf{0};t)$ is called the elasticity limit of the EPO. The oscillator defined by the equation of motion

$$\mathcal{L}\mathbf{X} \equiv \mathbf{M}\ddot{\mathbf{X}} + \mathbf{D}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{W} \tag{2}$$

without restriction on the response $\mathbf{X}(t)$, is called the associate linear oscillator (ALO) of the given EPO. Since the linear operator $\mathcal{L}$ commutes with the linear regression operator $\hat{E}$, it follows that the linear regression $\hat{E}[\mathbf{X} \mid \mathbf{Y}]$, $\mathbf{Y} = [\mathbf{X}^{\mathsf{T}}(0)\ \dot{\mathbf{X}}^{\mathsf{T}}(0)]^{\mathsf{T}}$, (i.e. the best linear prediction of $\mathbf{X}$ in terms of $\mathbf{Y}$) satisfies the equation of motion

$$\mathcal{L}\hat{E}[\mathbf{X} \mid \mathbf{Y}] = \hat{E}[W \mid \mathbf{Y}] = \mathbf{0} \tag{3}$$

This is the case also if the zero mean Gaussian white noise excitation $W(t)$ is generalized to a zero mean excitation that by integration becomes a process with independent increments. In case of Gaussian excitation the linear regression coincides with the conditional mean.

The linear regression $\hat{E}[\mathbf{X} \mid \mathbf{Y}]$ is the unique solution to the homogeneous equation of motion given the initial vectors of position $\mathbf{X}(0)$ and velocity $\dot{\mathbf{X}}(0)$. The standard second moment expression for the linear regression is (Ditlevsen & Madsen 1996) (p.64)

$$\hat{E}[\mathbf{X} \mid \mathbf{Y}] =$$

$$E[\mathbf{X}] + \mathrm{Cov}[\mathbf{X},\mathbf{Y}^{\mathsf{T}}]\,\mathrm{Cov}[\mathbf{Y},\mathbf{Y}^{\mathsf{T}}]^{-1}(\mathbf{Y}-E[\mathbf{Y}]) \tag{4}$$

Since $\mathbf{Y}$ can be set to an arbitrary vector, it follows from (4) that besides $E[\mathbf{X}]$ also $\mathrm{Cov}\left[\mathbf{X}, \mathbf{Y}^{\mathsf{T}}\right]$ satisfies the homogeneous equation of motion of the ALO.

The zero mean nonstationary Gaussian residual process $\mathbf{R} = \mathbf{X} - \hat{E}[\mathbf{X} \mid \mathbf{Y}]$ is identical to the random response of the ALO obtained as the solution to (2) for the initial conditions $\mathbf{X}(0) = \dot{\mathbf{X}}(0) = \mathbf{0}$. The residual covariance matrix is

$$\mathrm{Cov}[\mathbf{R}(s), \mathbf{R}^{\mathsf{T}}(t)] = \mathrm{Cov}[\mathbf{X}(s), \mathbf{X}^{\mathsf{T}}(t)]$$

$$-\mathrm{Cov}[\mathbf{X}(s), \mathbf{Y}^{\mathsf{T}}]\mathrm{Cov}[\mathbf{Y}, \mathbf{Y}^{\mathsf{T}}]^{-1}\mathrm{Cov}[\mathbf{Y}, \mathbf{X}^{\mathsf{T}}(t)] \tag{5}$$

These mathematical observations show that if the two response paths of the EPO and the ALO, respectively, generated by the same Gaussian white noise excitation history both pass any given point inside the elasticity domain with any given velocity, then the two response paths are identical as long as the one or the other of the paths is inside the elasticity domain. This identity of the two path pieces does not necessarily exist if the integrated excitation process has dependent increments.

The linear regression operator has the property

$$\hat{E}[\mathbf{X} \mid \mathbf{Z}] = \hat{E}\left[\hat{E}[\mathbf{X} \mid \mathbf{Y}]\middle|\mathbf{Z}\right] \tag{6}$$

where $\mathbf{Z}$ is any subvector of $\mathbf{Y}$. Thus the linear regression $\hat{E}[\mathbf{X} \mid \mathbf{Z}]$ is obtained by substituting (4) into (6) and using the linearity of the operator $\hat{E}$. This gives that $\hat{E}[\mathbf{X} \mid \mathbf{Z}]$ is obtained by setting

$$\mathbf{Y} := \hat{E}[\mathbf{Y} \mid \mathbf{Z}] \tag{7}$$

in (4). In other words, the linear regression $\hat{E}[\mathbf{X} \mid \mathbf{Z}]$ is obtained as the free ALO response (i.e. the solution to the homogeneous equation of motion) that corresponds to the vectorial initial condition (7).

In the following only the stationary ALO response of zero mean is considered. Applied to a stationary scalar process $X(t)$ the linear regression (4) becomes, noting that $\mathrm{Cov}[X(0), \dot{X}(0)] = 0$,

$$\hat{E}[X(t) \mid X(0), \dot{X}(0)]$$

$$= \frac{\mathrm{Cov}[X(0), X(t)]}{\mathrm{Var}[X(0)]} X(0) + \frac{\mathrm{Cov}[\dot{X}(0), X(t)]}{\mathrm{Var}[\dot{X}(0)]} \dot{X}(0)$$

$$= r(t)\, X(0) - \frac{\dot{r}(t)}{\lambda_2} \dot{X}(0) \tag{8}$$

where $r(t) = \mathrm{Corr}[X(0), X(t)]$ is the correlation coefficient function and λ_2 is the second spectral moment of the process. In particular $X(t)$ may be the ith component $X_i(t)$ of the ALO response. According to the remark after (7) the linear regression $\hat{E}[X_i(t) \mid X_i(0), \dot{X}_i(0)]$ is identical to the free ALO response corresponding to the initial conditions $X_1(0) := \hat{E}[X_1(0) \mid X_i(0), \dot{X}_i(0)]$, $\ldots$, $X_n(0) := \hat{E}[X_n(0) \mid X_i(0), \dot{X}_i(0)]$, $\dot{X}_1(0) := \hat{E}[\dot{X}_1(0) \mid X_i(0), \dot{X}_i(0)]$, $\ldots$, $\dot{X}_n(0) := \hat{E}[X_n(0) \mid X_i(0), \dot{X}_i(0)]$.

3 MODAL DECOMPOSITION

In the following the considerations are restricted to the case where the damping matrix $\mathbf{D}$ has such a form that a modal decomposition exists, i.e. there is a linear transformation $\mathbf{X}(t) = \mathbf{VT}(t)$ that diagonalizes all three matrices $\mathbf{M}, \mathbf{D}, \mathbf{K}$. Thus the coefficient matrices in

$$\mathbf{V}^\mathsf{T}\mathbf{MV}\ddot{\mathbf{T}} + \mathbf{V}^\mathsf{T}\mathbf{DV}\dot{\mathbf{T}} + \mathbf{V}^\mathsf{T}\mathbf{KVT} = \mathbf{V}^\mathsf{T}\mathbf{W} \qquad (9)$$

become the diagonal matrices $\mathbf{V}^\mathsf{T}\mathbf{MV} = \mathbf{I}$ (unit matrix), $\mathbf{V}^\mathsf{T}\mathbf{DV} = 2\underline{Z}\,\underline{\Omega}$, and $\mathbf{V}^\mathsf{T}\mathbf{KV} = \underline{\Omega}^2$, respectively. With standard notation we have $\underline{Z} = \{\zeta_i\delta_{ij}\}, \underline{\Omega} = \{\omega_i\delta_{ij}\}$, where δ_{ij} is Kronecker's delta, $\omega_1, \ldots, \omega_n$ are the modal undamped eigenfrequencies, and $\zeta_1, \ldots, \zeta_n$ are the modal damping ratios. Thus $\mathbf{V}$ is the common matrix of eigenvectors to the two eigenvalue problems $\mathbf{KV} = \mathbf{MV}\underline{\Omega}^2$ and $\mathbf{DV} = \mathbf{MV}\underline{Z}\,\underline{\Omega}$. Such a common eigenvector matrix exists if and only if the two matrices $\mathbf{M}^{-1}\mathbf{K}$ and $\mathbf{M}^{-1}\mathbf{D}$ commute by multiplication, i.e. if and only if $\mathbf{D}$ satisfies the condition $\mathbf{DM}^{-1}\mathbf{K} = \mathbf{KM}^{-1}\mathbf{D}$. Thus the rth modal equation of motion of the ALO gets the standard form $\ddot{T}_r + 2\zeta_r\omega_r\dot{T}_r + \omega^2 T_r = \mathbf{v}_r^T\mathbf{W}$ where $\mathbf{v}_r$ is the rth column in $\mathbf{V}$.

The rth modal impulse response function is $h_r(t) = \alpha_r^{-1}e^{-\zeta_r\omega_r t}\sin\alpha_r t,\ t \geq 0$, where $\alpha_r = \omega_r\sqrt{1-\zeta_r^2}$. Then,focusing in the following on the stationary ALO response, the relevant modal covariances become

$$\mathrm{Cov}[T_r(0), T_s(t)] =$$

$$\pi\,\mathbf{v}_r^\mathsf{T}\mathbf{G}\mathbf{v}_s\int_{-\infty}^{0} h_s(t-\tau_2)d\tau_2\int_{-\infty}^{t}\delta(\tau_1-\tau_2)h_r(-\tau_1)\,d\tau_1$$

$$= \pi\,\mathbf{v}_r^T\mathbf{G}\mathbf{v}_s\,e^{-\zeta_s\omega_s t}(a_{rs}\cos\alpha_s t + b_{rs}\sin\alpha_s t) \qquad (10)$$

where $\pi\mathbf{G}$ is the constant intensity matrix of the Gaussian white noise vector process $\mathbf{W}(t)$ ($\mathbf{G}$ is the one-sided spectral matrix), i.e. $\mathrm{Cov}[\mathbf{W}(s), \mathbf{W}^\mathsf{T}(t)] = \pi\mathbf{G}\,\delta(t-s)$. The coefficients a_{rs} and b_{rs} are given in Appendix 1. The covariance function matrix then is $\mathrm{Cov}[\mathbf{X}(0), \mathbf{X}^\mathsf{T}(t)] = \mathbf{V}\mathrm{Cov}[\mathbf{T}(0), \mathbf{T}^\mathsf{T}(t)]\mathbf{V}^\mathsf{T}$.

The dominating modal term $v_{ik}T_k(t)$ of the ith component $X_i(t)$ of $\mathbf{X}(t)$ is defined to be that for which $\forall r \in \{1, \ldots, n\} : v_{ik}^2\mathrm{Var}[T_k] \geq v_{ir}^2\mathrm{Var}[T_r]$. The corresponding angular frequency $\alpha^{(i)} = \alpha_k = \omega_k\sqrt{1-\zeta_k^2}$ is called the dominating angular frequency, and $2L_i = 2\pi/\alpha^{(i)}$ is called the dominating period for X_i.

4 ALO OUTCROSSINGS

The further restriction is made in the following that there is a representation $\mathbf{X}$ of the displacements of the ALO such that the elasticity domain is an axis parallel box that may depend on the plastic displacement history of the EPO. Moreover the elastic restoring force energy related to each displacement component of the ALO is assumed to be proportional to this displacement component solely. A structural example of this kind of system is a plane traverse shear frame with n heavy and rigid traverses connected by separated pairs of light columns where each pair of columns with respect to parallel translation has a symmetric linear elastic-plastic force-displacement constitutive relation. The component displacements in $\mathbf{X}$ are the relative displacements of the traverses.

First assume that all the component elasticity limits are infinite except for the ith component X_i. Make the preposition that the ALO is momentaneously changed into the EPO at the time of an outcrossing of the ALO-response through the elasticity limit. Then the initial conditions for the displacements of the EPO after the outcrossing are completely defined by the random vector of mass positions and mass velocities conditional on the event of the outcrossing. The joint distribution of this initial vector is known from the Gaussian response characteristics of the ALO. In particular, with the assumption that the ALO response is stationary, the distribution of the normalized outcrossing velocity $Z = \dot{X}_i/\sqrt{\lambda_{2i}}$ observed at the outcrossing point has the standard Rayleigh probability density $f_Z(x) = xe^{-x^2/2}$, $x \geq 0$, where λ_{2i} is the second spectral moment of X_i.

Let the time origin be at a point of upcrossing of the upper elasticity limit. After the upcrossing with velocity $\dot{X}_i(0)$ the ALO response component $X_i(t)$ in the mean follows the linear regression $\hat{E}[X_i(t)\,|\,X_i(0) = e_i,\ \dot{X}_i(0) = Z\sqrt{\lambda_{2i}}\,]$. The corresponding component of the residual Gaussian process starts out from the origin with zero velocity. For a lightly damped ALO the growth of the residual variance is sufficiently slow to be negligible relative to the linear regression value at the time of the occurrence of its first crest value after the upcrossing. Due to the damping of the free oscillations this crest value is the largest global value of the linear regression. The crest value $M_i\sqrt{\lambda_{0i}}$ can be obtained by numerical integration as a function of the normalized level $u_i = e_i/\sqrt{\lambda_{0i}}$, the normalized outcrossing velocity Z, the second spectral moment λ_{2i}, and the number of degrees of freedom n. For $n = 1$ the crest value can be obtained analytically from the Slepian model process representation $X_u(\tau) = [u\cos\tau + (u\alpha + Z\sqrt{1+\alpha^2})\sin\tau]e^{-\alpha\tau} + R(\tau)$ of the ALO response for $\tau > 0$ given an upcrossing of level u for $\tau = 0$, (Ditlevsen & Bognár 1993). Neglecting $\sqrt{1+\alpha^2}$, the maxima of the regression part are obtained for $\tan\tau \approx Z/(u + \alpha Z)$. The value of the square of the first maximum value of $X_u(\tau) - R(\tau)$

after the u-upcrossing then becomes

$$M^2 \approx [u^2+(\alpha u+Z)^2] \exp\left[-2\alpha \arctan\left(\frac{Z}{u+\alpha Z}\right)\right] \quad (11)$$

from which it is seen that $M^2 \to u^2 + Z^2$ as $\alpha \to 0$.

For other values of n an approximation of the form

$$\frac{M_i^2 - u_i^2}{Z^2} = \frac{a_i Z^{\beta_i} + b_i}{Z^{\beta_i} + b_i} \quad (12)$$

turns out to be applicable. In (12) the exponent $\beta_i \geq 0$, and the coefficients a_i, b_i may be fitted to be functions of u_i, λ_{2i}, and n. For modal damping of the ALO with $n > 1$ degrees of freedom and $\beta_i = 2$, some values of a_i and b_i that give applicable empirical fits are reported in (Randrup-Thomsen & Ditlevsen 1997). In the case $n = 1$ the ratio $(M^2 - u^2)/Z^2$ as defined by (11) is well approximated for $\alpha \leq 0.1$ by setting the right side of (12) to $(aZ + 2u)/(Z + 2u)$ with $a = 1 - 3.2\alpha$ for $u \geq 1$ and $a = 1 - 3.2\alpha + 7.0\alpha^2$ for $u = 0$.

For $u_i = 0$ the first crest is placed approximately a quarter of the dominating period $2L_i$ after the upcrossing point. Thus it follows from (8) that if $u_i = 0$, then $b_i \approx 0$ and

$$a_i \approx \frac{\dot{r}_i^2(L_i/2)}{\lambda_{2i}} \quad (13)$$

where $r_i(t) = \mathrm{Corr}[X_i(0), X_i(t)]$.

For the simplified approximation

$$\frac{M_i^2 - u_i^2}{Z^2} = a \quad (14)$$

[e.g. setting $a = (M_i^2 - u_i^2)/E[Z]^2$, $E[Z] = \sqrt{\pi/2}\,$] the maximal normalized elastic energy $E_i = \frac{1}{2} M_i^2$ per unit stiffness of the ALO displacement component X_i in a quarter cycle then gets the probability density

$$f_{E_i}(x) = \frac{1}{za} f_Z(z), \quad x \geq u_i^2/2 \quad (15)$$

where $z = (2x - u_i^2)/a$.

5 CONDITIONAL DISTRIBUTIONS

The density function (15) is conditional on an upcrossing of the normalized level u_i (or a downcrossing of a corresponding normalized lower elasticity limit) implying that $f_Z(z)$ is the standard Rayleigh density. However, for the purpose of deriving plastic displacement distributions for the EPO some further conditioning is necessary. An important situation is the following. At the moment when the EPO after an excursion outside the elasticity domain returns to the elasticity domain, it does it with a displacement of the ith component equal to the current plastic displacement plus an elastic displacement equal to some value $\pm\xi\sqrt{\lambda_{i0}}$, $\xi > 0$, ($\xi = e_i/\sqrt{\lambda_{i0}}$ for ideal plastic behavior), and with zero velocity of the ith component. With the initial conditions $X_j(0) := E[X_j(0) \mid X_i(0) = \xi, \dot{X}_i(0) = 0]$, $\dot{X}_j(0) := E[\dot{X}_j(0) \mid X_i(0) = \xi, \dot{X}_i(0) = 0]$, $j = 1, \ldots, n$, (for all $j \neq i$ these values are related to the stationary ALO response) for both the ALO response and the elastic part of the EPO response, the two responses become identical until the EPO response crosses outside the elasticity domain again. Due to symmetry it is sufficient to consider the situation with the initial plastic displacement $-\xi\sqrt{\lambda_{i0}}$. (The choice of setting the initial conditions for both the EPO and the ALO to the conditional expectations related to the stationary ALO response for all the components different from the ith is a simplification that will be discussed later). Setting the time origin at the occurrence of the maximal value of X_i after the start of the ALO with this initial condition, the start is approximately at the time $-L_i$. According to (8) (setting $\dot{X}(0) = 0$) and (5), the mean and the standard deviation of the normal distribution of $X_i(-L_i)/\sqrt{\lambda_{i0}}$ for $E_i = x \geq 0$ are $-\mu_i\sqrt{2x}$ and σ_i, respectively, where

$$\mu_i = -r_i(L_i) \quad (16)$$

$$\sigma_i^2 = \mathrm{Var}[R_i(L_i)/\sqrt{\lambda_{i0}}\,]$$

$$= 1 - r_i^2(L_i) - \frac{\lambda_{i0}}{\lambda_{i2}} \dot{r}_i^2(L_i) \quad (17)$$

Thus the conditional probability density

$$f_{E_i}(x \mid X_i(-L_i) = -\xi\sqrt{\lambda_{i0}}\,)$$

$$\propto f_{E_i}(x)\, f_{X_i}(-\xi\sqrt{\lambda_{i0}} \mid M_i = \sqrt{2x}\,)$$

$$\propto f_{E_i}(x)\, \varphi\Big(\frac{\xi - \mu_i\sqrt{2x}}{\sigma_i}\Big), \; x \geq u_i^2/2 \quad (18)$$

is obtained by use of Bayes' formula, with $\varphi(x) = e^{-x^2/2}/\sqrt{2\pi}$ being the standard normal density function. Similarly the conditional density $f_{E_i}(x \mid X_i(-L_i) \in [-\xi\sqrt{\lambda_{i0}}, \xi\sqrt{\lambda_{i0}}\,])$ is obtained as

$$f_{E_i}(x \mid X_i(-L_i) \in [-\xi\sqrt{\lambda_{i0}}, \xi\sqrt{\lambda_{i0}}\,]) \propto f_{E_i}(x)$$

$$\cdot P[X_i(-L_i) \in [-\xi\sqrt{\lambda_{i0}}, \xi\sqrt{\lambda_{i0}}\,] \mid M_i = \sqrt{2x}\,]$$

$$\approx f_{E_i}(x)\, P[X_i(-L_i) \geq -\xi\sqrt{\lambda_{i0}} \mid M_i = \sqrt{2x}\,]$$

$$\propto f_{E_i}(x)\, \Phi\Big(\frac{\xi - \mu_i\sqrt{2x}}{\sigma_i}\Big), \quad x \geq u_i^2/2 \quad (19)$$

where $\Phi(x) = \int_{-\infty}^{x} \varphi(u)\,du$ is the standard normal distribution function. The approximation obtained by replacing the upper interval bound $\xi\sqrt{\lambda_{i0}}$ by ∞ is accurate when ξ is just some few times larger than σ_i. The conditional distribution functions corresponding to (18) and (19) are given for the simple approximation (15) in Appendix 2.

Simulation of realizations of E_i according to (18) or (19) is simplified considerably if (14) is assumed to be valid. An approximate correction to such a realization can be computed by substituting $Z = \sqrt{(2E_i - u^2)/a}$ in (11) or (12) to give a value of M_i^2. From this value the corrected realization $E_i = M_i^2/2$ is next obtained.

6 PLASTIC CLUMP DEFINITION

Each time the EPO has an excursion to the plastic domain there is an accumulation of plastic displacement and possibly also a change of the extension of the elastic domain. To take this into account the notation $\mathcal{E}_m(\mathbf{0}) = [-e_{mi}, e_{mi}]$ is used for the symmetrized elastic domain after the mth excursion to the plastic domain. The current symmetrized elastic domain is defined to be the largest symmetrical subset of the current elastic domain. Elastic-ideal plastic behavior is a special case characterized by the property that the elastic domain is invariant. The variable $\xi\sqrt{\lambda_{i0}}$ is introduced above in consideration of possible asymmetry of the elastic domain.

A clump of plastic displacements of the EPO is defined as follows. An outcrossing out of $\mathcal{E}_m(\mathbf{0}) = [-e_{mi}, e_{mi}]$ is said to be the first in a clump if the ALO response vector X_i is in the interval $[-e_{mi}, e_{mi}]$ at the time point placed L_i before the time of occurrence of the first crest of X_i after the outcrossing. It is also said to be the last in this clump if the ALO response vector X_i is in the interval $[-e_{(m+1)\,i}, e_{(m+1)\,i}]$ at the time point placed L_i after a start from zero velocity component $\dot{X}_i$ from a position with $X_i = \pm\xi_{m+1}\sqrt{\lambda_{i0}}$ corresponding to an EPO return from an excursion outside the symmetrized elastic domain. According to this definition it is not excluded that the corresponding plastic displacement becomes zero. If the first outcrossing in the clump is not the last in the clump, there are one or more following outcrossings from the symmetrized elastic domain with the last being characterized in the same way as for the first outcrossing also being the last outcrossing. A clump of plastic displacements of the EPO is then defined as the set of plastic displacements resulting from the outcrossings in a clump. The total absolute plastic displacement from a clump of plastic displacements can only be zero if the clump has no more than one outcrossing.

7 KARNOPP-SCHARTON

When an outcrossing of the ALO response through the elasticity limit occurs, the EPO starts to get plastic displacements that increase in size as long as the velocity vector has a positive component along the outwardly directed normal vector to the elasticity limit (which according to the standard theory of plasticity is translated along the normal with this velocity component). The first time the velocity component reaches zero, the plastic displacement increase stops, and the EPO returns to the ALO behavior. However, for usual plasticity and hysteresis models the minor fluctuations of the normal velocity component to the negative side give only small contributions to the total plastic displacement obtained within the actual quarter cycle of the EPO displacement defined by the dominating period. Such minor fluctuations are caused both by the white noise excitation and by the non-dominating terms in the modal superposition. The effect of these fluctuations is suppressed by use of the approximation principle of Karnopp and Scharton (Karnopp & Scharton 1966). According to this principle the maximal elastic energy E_i in a quarter cycle of the ALO is set equal to the quarter cycle work of the EPO. Thereby an estimate of the total plastic displacement in a quarter cycle is obtained. The Karnopp-Scharton principle is an approximation due to the mentioned velocity fluctuations to the negative side and, if $n > 1$, also because a part of the elastic excess energy may transfer to kinetic energy of the velocity components tangential to the elasticity limit and vice versa. In studied examples (Ditlevsen & Randrup-Thomsen 1996, Randrup-Thomsen 1997) this energy transfer is only a small fraction of E_i.

In case of ideal plasticity the Karnopp-Scharton principle gives the absolute plastic displacement $D = \max\{0, (E_i - u_i^2/2)/u_i\}$. If D is the first absolute plastic displacement D_1 in the first occurring clump, then the probability distribution of D_1 is directly obtained from (19) with $\xi = u_i$. If the clump is not the first occurring, then the distribution (19) is an approximation of increasing accuracy for increasing time distance to the previous clump.

If D is the second absolute plastic displacement D_2 or any of the following absolute plastic displacements in a clump, then the conditional probability distribution of D_j given $D_j > 0$, $j > 1$, is obtained from (18) with $\xi = u_i$. The probability $p = P(D_j > 0) = P(E_i > u_i^2/2)$ is obtained from (18) setting $u_i = 0$ in (14) and (15). The random number $N \geq 1$ of plastic displacements in the clump is thus defined by the event $D_1 > 0$, $D_2 > 0$, $\dots$, $D_N > 0$, $D_{N+1} = 0$. This number N is called the clump size, and for a one degree of freedom oscillator ($n = 1$) it is directly seen that N has a geometric distribution, i.e. $P(N = \nu) = p^{\nu-1}(1-p)$, $\nu \in \mathbb{N}$.

The total absolute plastic displacement in a clump is $D_{abs} = D_1 + \ldots + D_N$, and the total net plastic displacement is $D_{net} = \pm(D_1 - D_2 + \ldots - (-1)^N D_N)$ with equal probability on the two signs. These results are for a single degree of freedom EPO confirmed by direct response simulations as reported in (Ditlevsen & Bognár 1993). Moreover, it is shown in (Ditlevsen & Bognár 1993) that the distributions of the independent random variables D_1 and D_2 (and the following plastic displacements in a random clump) can be replaced by two approximating exponential distributions to allow an analytical derivation of quite accurate approximate expressions for the distributions of D_{abs} and D_{net}. The approximate distribution of D_{abs} is a mixture of two exponential distributions and the approximate distribution of D_{net} is a mixture of two Laplace distributions. For both distributions all parameters are given by closed form expressions.

Gaussian white noise excited linear elastic-ideal plastic EPOs of more than one degree of freedom have been successfully subject to the same kind of simple plastic displacement distribution predictions (Ditlevsen & Randrup-Thomsen 1996, Randrup-Thomsen & Ditlevsen 1997, Randrup-Thomsen 1997). The considered EPOs are traverse frames of type as described in the beginning of Section 4 with linear elastic-ideal plastic connections between the two top traverses and with all other connections being linear elastic, and with the excitation acting on the bottom traverse.

For a general hysteretic constitutive behavior of the connections between the discrete masses the determination of the plastic displacement distributions by use of the Slepian process method must necessarily be by numerical simulations based on the distributions (18) and (19). A three degree of freedom traverse frame with linear elastic-hardening plastic constitutive behavior of the top column connection, and for which experimental data are available, have been subject to detailed calculations with good agreement between measured and calculated distributions of D_{net} and D_{abs} (Randrup-Thomsen & Ditlevsen 1998, Ditlevsen & Tarp-Johansen 1997).

8 CRITICAL OBJECTIONS

It is right to be critical with respect to some neglected details in the analytical derivation of these approximate plastic displacement distributions for EPOs of more than one degree of freedom, essentially treating the EPOs as if they are of one degree of freedom. As mentioned above, a point of criticism is that the Karnopp-Scharton principle does not take the possible transfer of energy between the plastic part and the elastic part of the EPO into account. Another point concerns the neglect of the effect of the actual deviations from the conditional mean values of the positions and velocities of the components X_j, $j \neq i$, at each crossing of X_i out of the elastic domain.

However, more correct numerical calculations can be made by use of the Slepian model process technique including all components of $\mathbf{X}$ for simulating a suitably large sample of clumps of plastic displacements. Without considering the energy transfer problem, each maximum M_i of the ALO response component X_i above the elasticity limit is then distributed conditional on 1) X_i being equal to one of the elasticity limits, 2) $\dot{X}_i = 0$, and 3) the current values of X_j and $\dot{X}_j$, $j \neq i$, half the dominating period $2L_i$ before the time of occurrence of the maximum of X_i. The values of X_j and $\dot{X}_j$, $j \neq i$ occurring simultaneously with this maximum are generated from the corresponding conditional multidimensional Gaussian distribution. Using the obtained values as conditioning values, new values can be generated for assignment to the time L_i forward. This procedure is repeated until the termination of the clump according to definition. Only the conditioning related to the start of the clump is different. The conditioning event is then simply that X_i is inside the symmetrized elasticity interval.

The effect of the splitting into plastic work and kinetic energy of the available excess energy of the ALO during the plastic phase of the restoring force can be approximately evaluated in the following way. The duration of the plastic displacement growth until the maximum is reached can for any given outcrossing velocity be approximately assessed from the linear regression of the ALO response component X_i. The integral from the time of the upcrossing to the time of the maximum of the difference between the plastic restoring force and the elastic restoring force as given approximately by the linear regression then defines a correcting impulse to be applied to those masses that are directly subject to the plastic restoring forces. The effect is that the ALO velocities of these masses make jumps corresponding to the increase of momentum from the impulses. These jumps cause corresponding jumps in the kinetic energy of the masses. The elastic excess energy minus this jump in kinetic energy (that may be positive or negative) is thus available for the plastic work (Ditlevsen & Randrup-Thomsen 1996, Randrup-Thomsen & Ditlevsen 1997). Such corrections can in an obvious way be directly built into the above described procedure for simulating samples of clumps of plastic displacements by the Slepian model process method.

Example calculations reported in (Ditlevsen & Randrup-Thomsen 1996, Randrup-Thomsen & Ditlevsen 1997, Randrup-Thomsen 1997) show that the energy transfer correction has only a very small effect on the plastic displacement distributions. The correction may increase or decrease the size of the plastic displacement. In absolute value the correction increases for decreasing elasticity limit and for increasing number of degrees of freedom. The influence of the conditioning on the current values of X_j and $\dot{X}_j$, $j \neq i$, is more visible but also small. It shows up as the occurrence of a larger number of small size plastic displacements than obtained without this conditioning. Also this effect increases for decreasing elasticity limit and increasing number of degrees of freedom.

To test the Slepian modeling technique itself, direct EPO response simulations have been made. Due to the long time distances between the excursions to the plastic domain such calculations are much more time consuming than the Slepian model calculations. For example, for $n = 4$, time factors of about 5, 15, 200 for $u_i = 1, 2, 3$, respectively, have been experienced (Randrup-Thomsen 1997). These direct response simulations give results that nicely fit with the results from the Slepian process method.

9 CLUMP DISTANCES

Based on the level crossing properties of the Cramér-Leadbetter envelope (Cramér & Leadbetter 1967) of the response of the one degree of freedom ALO it is shown in (Ditlevsen & Bognár 1993) that the distance T from the end of a clump to the start of the next clump for the one degree of freedom EPO can be well approximated as a random variable with an exponential distribution shifted by half the period L and having the expectation

$$E[T] - L = L + \sqrt{\frac{2\pi\lambda_0}{\lambda_2}}\,\frac{e^{\frac{1}{2}u^2} - 1}{u\,\delta\,Q(u,\delta)} \tag{20}$$

$$Q(u,\delta) \approx 1 -$$

$$2u\int_0^1 \varphi(ux)\Big[1 - \sqrt{2\pi}\frac{2\Phi[\gamma u(1-x^2)] - 1}{2\gamma u(1-x^2)}\Big]dx \tag{21}$$

$$\gamma = \frac{\pi\delta}{2\sqrt{1-\delta^2}} \tag{22}$$

$$\delta^2 = 1 - \frac{\lambda_1^2}{\lambda_0\lambda_2} \tag{23}$$

where $u = e/\sqrt{\lambda_0}$. The function $Q(u,\delta)$ is the long run fraction of so-called qualified excursions outside the interval $[-e, e]$, that is, envelope excursions for which there are outcrossings of the ALO response during the excursion (Ditlevsen & Lindgren 1988, Ditlevsen 1994). The number $\delta > 0$ is the so-called spectral width parameter of the ALO response. For one degree of freedom δ becomes (Vanmarcke 1983) p. 180.

$$\delta = \sqrt{1 - \frac{1}{1-\zeta^2}\Big(1 - \frac{1}{\pi}\arctan\frac{2\zeta\sqrt{1-\zeta^2}}{1-2\zeta^2}\Big)^2} \tag{24}$$

For the ideal EPO it is also shown in (Ditlevsen & Bognár 1993) that the clump duration can be approximated by an exponential distribution with mean $-L/\log P(D > 0)$. For the non-ideal EPO the mean duration $E[N]L$ of a clump of plastic displacements is estimated directly from the simulated sequences $D_1, \ldots, D_N$ of absolute plastic displacements.

Simulation checking shows that both (20) for the mean time $E[T]$ between clump occurrences and the exponential distribution assumption apply with good accuracy also for an EPO of more than one degree of freedom (Randrup-Thomsen 1997). This can be used to calculate the growth rates of relevant statistics of the accumulated absolute and net plastic displacements, respectively. To obtain the second spectral moment λ_2 and the spectral width parameter δ, advantage can be taken of the modal decomposition $\mathbf{X}(t) = \mathbf{VT}(t)$ of the response as derived in Section 3. When the eigenfrequencies are well separated, the covariances $\text{Cov}[T_r(t), T_s(t)]$, $r \neq s$, are sufficiently small compared to the variances $\text{Var}[T_r(t)]$ to allow the approximation of neglecting their contributions. Thus $\text{Cov}[X_i(0), X_i(t)] \approx \sum_{r=1}^n v_{ir}^2 \text{Cov}[T_r(0), T_r(t)]$ implying that the mth spectral moment corresponding to the ALO response component $X_i(t)$ becomes approximated as

$$\lambda_m \approx \sum_{r=1}^{n} v_{ir}^2 \lambda_{mr} \tag{25}$$

where λ_{mr} is the mth spectral moment corresponding to $T_r(t)$. According to (23) we get

$$1 - \delta^2 = \frac{(\sum_{r=1}^n v_{ir}^2\lambda_{1r})^2}{\sum_{r=1}^n v_{ir}^2\lambda_{0r}\sum_{r=1}^n v_{ir}^2\lambda_{2r}}$$

$$= \frac{(\sum_{r=1}^n v_{ir}^2\lambda_{0r}\omega_r\sqrt{1-\delta_r^2})^2}{\sum_{r=1}^n v_{ir}^2\lambda_{0r}\sum_{r=1}^n v_{ir}^2\omega_r^2\lambda_{0r}} \tag{26}$$

using that $\lambda_{2r} = \lambda_{0r}\omega_r^2$ for the modal response $T_r(t)$, (Ditlevsen & Bognár 1993), and $\lambda_{1r}^2 = \lambda_{0r}\lambda_{2r}(1 - \delta_r^2)$, where δ_r^2 is the spectral width parameter of the modal response $T_r(t)$. Finally (Ditlevsen & Bognár 1993)

$$\lambda_{0r} = \pi\, \mathbf{v}_r^T \mathbf{G} \mathbf{v}_r\, a_{rr} = \pi\, \mathbf{v}_r^T \mathbf{G} \mathbf{v}_r\, \frac{1}{4\omega_r^3 \zeta_r} \tag{27}$$

The exponential distribution approximation is less accurate for low crossing levels. Due to the oscillatory nature of the response and the frequent crossings of low levels there is a tendency of having concentrated probability densities around time points separated by half a period (for symmetric double barrier). For increasing time the corresponding step like nature of the distribution function fades out and asymptotically approaches an exponential upper tail.

The assumption of Section 4 that only the ith component gets plastic displacements can be relaxed if the dimensionless elasticity limits u_j, $j = 1, \ldots, n$ are reasonably large (Randrup-Thomsen 1997). The requirement is that the clumps of yielding with suitably high probability occur separated in time from each other. Then the n plastic displacement processes can be joined as the components in the vectorial plastic displacement process of the n degree of freedom EPO.

10 INITIAL CONDITIONS AND SLEPIAN SIMULATION

The initial condition of the EPO compatible to the Slepian modeling of the distribution of the first plastic displacement in a clump is defined by a modification of the distribution of the stationary response of the ALO at an arbitrary point in time, e.g. at $t = 0$. If the displacement of the ALO at time $t = 0$ is outside the elasticity domain, then the initial elastic displacement component $X_i(0)$ of the EPO is put equal to the relevant elasticity limit. If in addition the velocity component $\dot{X}_i(0)$ is directed away from the equilibrium position, the EPO is given an instantaneous plastic displacement at $t = 0$ corresponding to setting the plastic work equal to the excess elastic energy of the ALO. Otherwise, if the ALO displacement is inside the elasticity limits, the initial elastic displacement of the EPO is the same as that for the ALO.

The EPO may have any initial plastic displacement that may be included in the definition of the EPO and the corresponding ALO. Thus it is sufficient to assume that the initial plastic displacement is zero.

This initial condition is used to start a direct numerical integration simulation of the EPO response, e.g. as a vectorial Markov sequence with suitably small time steps (Ditlevsen, Tarp-Johansen, & Randrup-Thomsen 1998). The corresponding initial condition for the Slepian simulation is obtained by considering that the stationary ALO response alternates between clumps of excursions outside the elasticity domain of the EPO and separating intervals with oscillations within the elasticity domain. If the time origin is chosen at random on the time axis, then the probability p that the origin falls within a clump of excursions is the ratio of the mean clump duration to the mean duration from the start of a clump to the start of the following clump. A simple approximation is obtained by setting $p = Q(u, \delta) \exp(-u^2/2)$ which is the probability that the Cramér-Leadbetter envelope of the ALO response has a qualified excursion outside the normalized interval $[-u, u]$ at any given point in time. Given that the origin falls within a clump, then the first plastic displacement after $t = 0$ is generated by use of the standard Rayleigh density and not by use of the conditional density (19). This initial plastic displacement is allocated as a jump of the plastic displacement to a time point chosen at random in the interval $]0, L/2]$. The sign of the jump is chosen at random. Thereafter the simulation proceeds as explained in Section 7. Within a clump the successive displacement jumps are allocated with distance L such that the duration of the clump becomes NL, where N is the random clump size.

The distance from the end of a clump to the start of the next clump is simulated as follows. The earlier mentioned step-like behavior of the distribution function of the duration between the end of a clump to the start of the next clump is to some extendtaken into account by use of continued simulation of the consecutive alternating troughs/crests and crests/troughs of the ALO response during a suitable number n of half periods L (e.g. $n = 4$). For each half period L this is done by conditioning both on the crossing of the level $u = 0$ and the simulated trough/crest displacement ξ at $L/2$ before the crossing of $u = 0$, that is, by generating a value of M from the distribution (34). If a crossing out of the elasticity domain occurs within these n half-periods, a new clump starts. If no outcrossing occurs before time $(n+1)L$ after the termination of the clump, then a realization of the eksponentially distributed random variable T with the expectation (20) is generated and added to $(n+1)L$ to give the time to the next clump. The sign of the next outcrossing level is then chosen to $\pm(-1)^{[n+1+T/L]}$, where $[x]$ means integer part of x. This sign rule interpolates between correct choice for small time distances in the mean (low levels) to about random choice for large time distances in the mean (high levels).

If the origin falls within a separating interval of elastic displacements, the duration to the occurrence of the first clump of plastic displacements can be assumed to be exponentially distributed with the expectation (20) using that the exponential distribution has the property of being memoryless, i.e. the time from the origin to the first clump has the same exponential distribution as the time between two consecutive clumps. Each response starting in a clump is assigned the weight p. Otherwise the weight $1-p$ is assigned to the response.

11 EXAMPLE

The mechanical system behind the following example is a portal frame with a rigid traverse on slender linear elastic-ideal plastic columns represented by a yield hinge model. The frame is subject to a vertical force that may cause column tension, giving plastic hardening behavior of the EPO, or column compression, giving plastic softening behavior of the EPO, both effects due to the bending moments from the horizontally displaced vertical force. The frame is subject to horizontal Gaussian white noise excitation. Details are given in (Ditlevsen, Tarp-Johansen, & Randrup-Thomsen 1998).

The ALO corresponding to the EPO of one degree of freedom defined in the following is used to non-dimensionalize time and displacement. In the following τ is the non-dimensional time defined such that the free damped ALO gets the oscillation period 2π. The displacement of the ALO is made non-dimensional such that the standard deviation of the stationary response of the ALO is 1. The plasticity properties of the EPO are defined by the scaling factor $A > 0$ of the elasticity limit, the parameter $\lambda > -\pi$ (linear softening for $\lambda < 0$, ideal plasticity for $\lambda = 0$, linear hardening for $\lambda > 0$) and the function $g(\lambda)$ defined by

$$g(\lambda) = \begin{cases} \dfrac{\lambda^3 \sinh\lambda}{12\,(2 - 2\cosh\lambda + \lambda\sinh\lambda)} & \text{for } \lambda \geq 0 \\ \dfrac{\lambda^3 \sin\lambda}{12\,(2 - 2\cos\lambda - \lambda\sin\lambda)} & \text{for } \lambda \leq 0 \end{cases} \qquad (28)$$

$$\sim 1 + \frac{1}{10}\lambda|\lambda| \text{ as } \lambda \to 0$$

The viscous damping ratio ζ of the ALO (29) enters through the parameter $\alpha = \zeta/\sqrt{g(\lambda) - \zeta^2}$. The EPO is then defined by the following equation of motion:

Elastic domain: (29)

$$\ddot{X}(\tau) + 2\alpha\dot{X}(\tau) + (1+\alpha^2)[X(\tau) - X_{i-1}] = W(\tau)$$

Plastic domain: (30)

$$\frac{\lambda|\lambda|}{12\,g(\lambda)}(1+\alpha^2)[X(\tau) + \text{sign}(\dot{X})\frac{12\,A}{\lambda|\lambda|}]$$

$$+\ddot{X}(\tau) + 2\alpha\dot{X}(\tau) = W(\tau)$$

$$\text{for} \quad X\,\text{sign}(\dot{X}) > X_{p\,(i-1)}\,\text{sign}(\dot{X}) + Ah(\lambda)$$

where $h(\lambda) = [g(\lambda) - |\lambda|\lambda/12]^{-1}$. Moreover

$$X_0 = 0, \quad X_i = X_{pi}\Big(1 - \frac{\lambda|\lambda|}{12\,g(\lambda)}\Big), \quad i \in \mathbb{N} \qquad (31)$$

where X_{pi} is the non-dimensional accumulated plastic displacement after the ith visit to the plastic domain. The external excitation $W(\tau)$ is Gaussian white noise with intensity $4\alpha(1+\alpha^2)$, that is, with $\text{Cov}[W(\tau_1), W(\tau_2)] = 4\alpha(1+\alpha^2)\delta(\tau_1 - \tau_2)$.

The non-dimensional displacement-force diagram is shown in Fig. 1. The diagram is well suited for illustrating the definition of the symmetrized elastic domain and also the definition used herein for a clump of plastic displacements. A crossing into the plastic domain marks the start of a clump of plastic displacements if the local extreme of the response directly before the outcrossing is in the local symmetrized elastic domain. The clump terminates when for the first time after the start an incrossing to the elastic domain is followed by a local extreme which is inside the local symmetrized elastic domain.

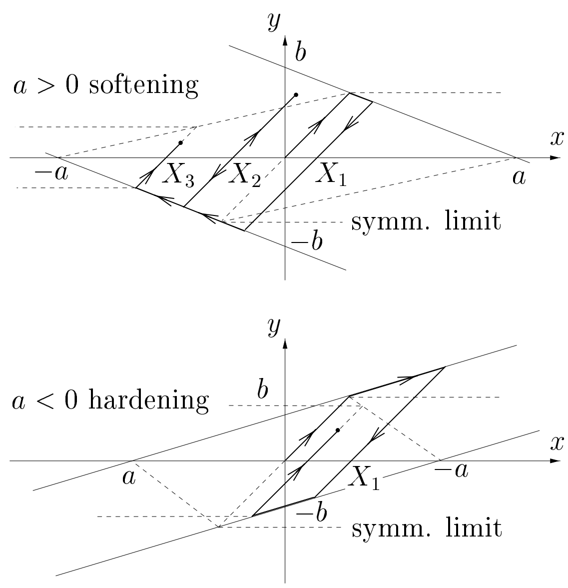

Figure 1. Illustration of clump definition as interpreted on the idealized dimensionless displacement-restoring force diagram corresponding to constant normal force in the frame columns.

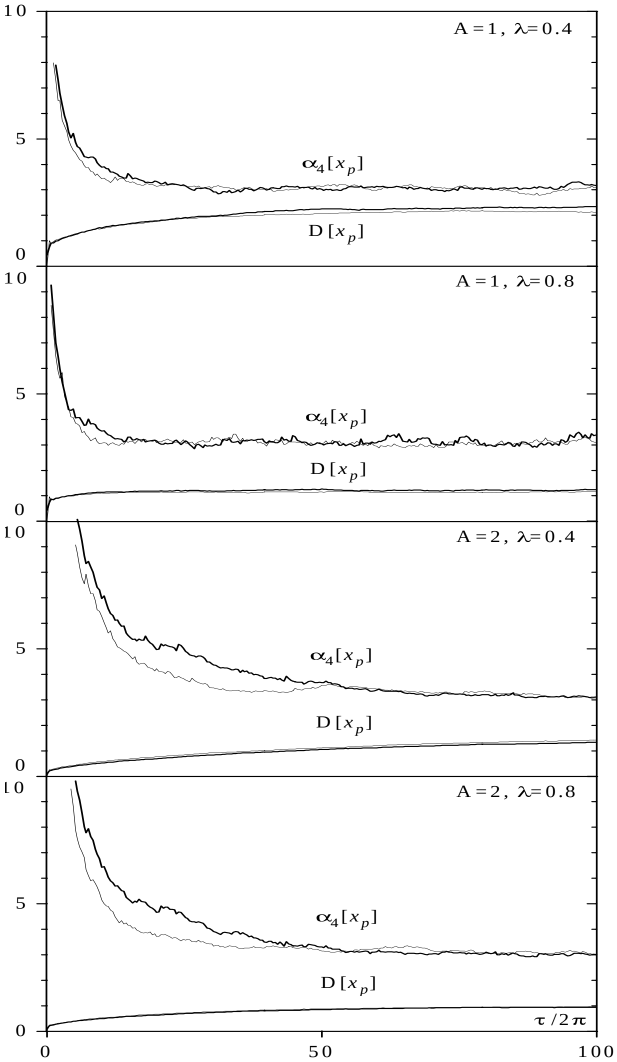

Figure 2. Standard deviation and kurtosis as function of time of the plastic displacement for EPO with hardening plasticity. Thick curves: direct simulation. Thin curves: Slepian simulation.

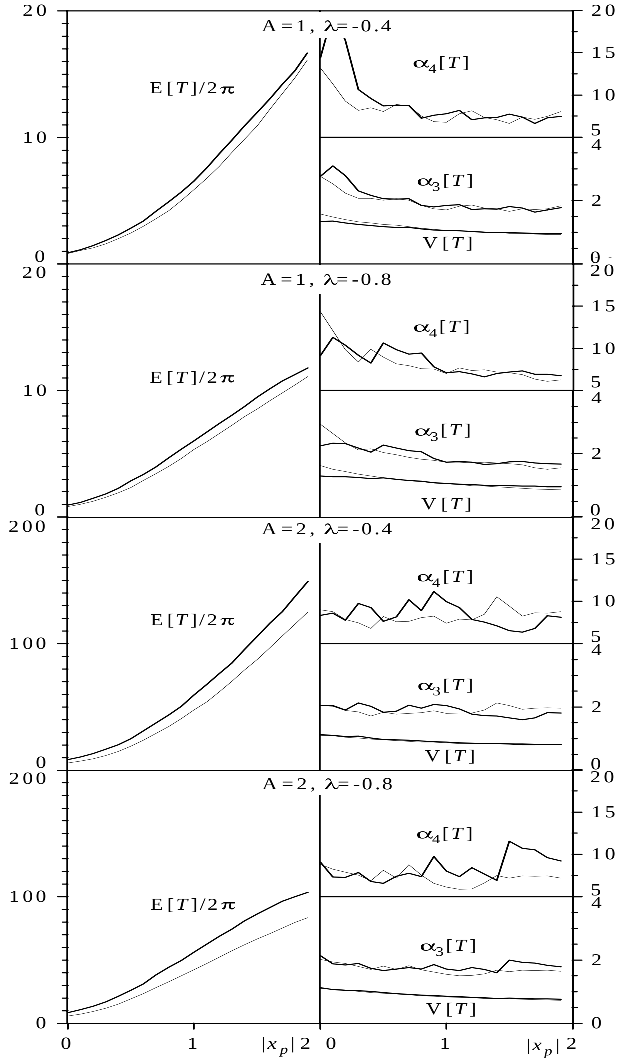

Figure 3. Statistics for first passage time of plastic displacement level for EPO with softening plasticity. Thin curves: Slepian simulation. Thick curves: direct simulation.

Figure 2 shows the time development of the standard deviation and the kurtosis of the plastic displacement for the damping parameter value $\alpha = 0.05$, two levels of hardening plasticity defined by $\lambda = 0.4, 0.8$ and two values $A = 1, 2$ of the yield level. Due to the symmetry of the EPO the mean function and skewness function are zero. The thin curves are obtained by fast Slepian simulation while the thick curves are from slow direct response simulation. The convergence to a constant value of the standard deviation for increasing time is seen to be the fastest for the largest value of λ, as it should be expected from the fact that as $\lambda \to 0$ the response approaches a non-stationary response with variance function that asymptotically increases proportional to time. The kurtosis approaches the value 3 consistent with the central limit theorem according to which the distribution of the accumulated plastic displacement approaches the normal distribution.

For the same damping parameter and yield levels, but with the values $\lambda = -0.4, -0.8$ corresponding to softening plasticity, the thin curves in Fig. 3 show the estimated mean time to first passage of a given plastic displacement level as function of this level.

Figure 3 shows the estimates of the coefficient of variation, the skewness, and the kurtosis of the first passage time. They are in the same figure compared with results (thick curves) obtained by direct simulation (Ditlevsen, Tarp-Johansen, & Randrup-Thomsen 1998). It is seen that all three relative statistics, i.e. coefficient of variation, skewness, and kurtosis, are insensitive to the degree of softening λ and asymptotically for increasing plastic displacement level $|x_p|$ also to the yield level A. For the plastic displacement level $|x_p|$ within a range from 1 to 2, the values of these relative statistics are $V \approx 0.9$, $\alpha_3 \approx 1.6$, and $\alpha_4 \approx 7$. These values are close to those of a random variable of the form T^ν where T has an exponential distribution and $\nu = 1.15$, that is, a distribution of Weibull type.

The direct simulation is made as a vectorial Markov sequence as explained in details in (Ditlevsen, Tarp-Johansen, & Randrup-Thomsen 1998). The simulated sample functions are thereby restricted to the discrete set $h, 2h, \ldots$ of time points for a suitably small value of h. The computation time gain factor depends on the choice of parameters and on the choice of time step in the direct simulation. In the present example the gain factor ranges from 30 to 250 for time steps between one hundredth and one thousandth of a period of the oscillator. In accordance with the nonlinear dependence of the mean interclump duration on the yield level the gain factor will increase rapidly with increasing u.

ACKNOWLEDGMENTS

This work was financially supported by the Danish Technical Research Council.

REFERENCES

Cramér, H. & M. Leadbetter (1967). *Stationary and Related Stochastic Processes.* New York: Wiley.

Ditlevsen, O. (1986). Elasto-plastic oscillator with gaussian excitation. *J. Engrg. Mech., ASCE* 112:386–406.

Ditlevsen, O. (1991). Gaussian excited elasto-plastic oscillator with rare visits to plastic domain. *J. Sound Vib.* 145:443–456.

Ditlevsen, O. (1994). Qualified envelope excursions out of an interval for stationary narrow-band gaussian processes.*Journal of Sound and Vibration* 173(3):309–327.

Ditlevsen, O. & L. Bognár (1993). Plastic displacement distributions of the gaussian white noise excited elasto-plastic oscillator. *Probabilistic Engineering Mechanics* 8:209–231.

Ditlevsen, O. & G. Lindgren (1988). Empty envelope excursions in stationary gaussian processes*Journal of Sound and Vibration* 122(3):571–587.

Ditlevsen, O. & H. Madsen (1996). *Structural Reliability Methods.* Chichester UK: Wiley.

Ditlevsen, O. & S. Randrup-Thomsen (1996). Gaussian white noise excited elasto-plastic oscillator of several degrees of freedom. In S. Krenk & A. Naess (eds.), *Proc. of IUTAM Symposium on Advances in Nonlinear Stochastic Mechanics, Trondheim 1995*, Dordrecht, Kluwer Academic, pp. 127–142.

Ditlevsen, O. & N. Tarp-Johansen (1997). White noise excited non-ideal elasto-plastic oscillator. *Acta Mechanica* 125:31–48.

Ditlevsen, O., N. Tarp-Johansen, & S. Randrup-Thomsen (1998). Elasto-plastic shear frame under both vertical and horizontal gaussian excitation. In B. Spencer & E. Johnson (eds.), *Proc. of Fourth International Conference on Stochastic Structural Dynamics, Notre Dame, Indiana, Aug. 6–8, 1998*, Rotterdam, Netherlands. Balkema.

Kac, M. & D. Slepian (1959). Large excursions of gaussian processes. *Annals of Mathematical Statistics* 30:1215–1228.

Karnopp, D. & T. Scharton (1966). Plastic deformation in random vibration. *J. Acoust. Soc. Amer.* 39:1154–1161.

Lin, Y. & G. Cai (1995). *Probabilistic Structural Dynamics.* New York: McGraw-Hill.

M. R. Leadbetter, G. L. & H. Rootzen (1983). *Extremes and Related Properties of Random Sequences and Processes.* New York: Springer-Verlag.

Randrup-Thomsen, S. (1997). *Analysis of the white noise excited elasto-plastic oscillator of several degrees of freedom.* Ph. D. thesis, Department of Structural Engineering and Materials, Technical University of Denmark, Building 118, DK-2800 Lyngby, Denmark. Published as No. 28 in Series R.

Randrup-Thomsen, S. & O. Ditlevsen (1997). One-floor building as elasto-plastic oscillator subject to and interacting with gaussian base motion. *Probabilistic Engineering Mechanics* 12:49–56.

Randrup-Thomsen, S. & O. Ditlevsen (1998). Experiments with elasto-plastic oscillator. *Probabilistic Engineering Mechanics.* in press.

Roberts, J. (1980). The yielding behaviour of a randomly excited elasto-plastic structure. *J. Sound Vib.* 72:71–85.

Soong, T. & M. Grigoriu (1993). *Random Vibration of Mechanical and Structural Systems.* Englewood Cliffs, New Jersey: Prentice Hall.

Vanmarcke, E. (1983). *Random Fields: Analysis and Synthesis.* Cambridge, Massachusetts: MIT Press.

APPENDIX 1

$$a_{rs} = \frac{2(\zeta_r\omega_r + \zeta_s\omega_s)}{(\omega_r^2 + \omega_s^2 + 2\omega_r\omega_s\zeta_r\zeta_s)^2 - 4\omega_r^2\omega_s^2(1-\zeta_r^2)(1-\zeta_s^2)} \tag{32}$$

$$b_{rs} = \frac{[2\zeta_s\omega_s(\omega_r\zeta_r + \omega_s\zeta_s) + \omega_r^2 - \omega_s^2]/(\omega_s\sqrt{1-\zeta_s^2})}{(\omega_r^2 + \omega_s^2 + 2\omega_r\omega_s\zeta_r\zeta_s)^2 - 4\omega_r^2\omega_s^2(1-\zeta_r^2)(1-\zeta_s^2)} \tag{33}$$

APPENDIX 2

For the approximation $M^2 - u^2 = aZ^2$ the complementary conditional distribution function of the quarter cycle elastic energy E of the component X of the ALO response given that $X(-L) = -\xi\sqrt{\lambda_0}$ is:

$$x \geq \frac{u^2}{2}: \quad 1 - F_E(x \,|\, X(-L) = -\xi\sqrt{\lambda_0}) =$$

$$\frac{\varphi\Big(\dfrac{\kappa^2\sqrt{2x} - a\mu\xi}{\sigma\kappa\sqrt{a}}\Big) + \dfrac{a\mu\xi}{\sigma\kappa\sqrt{a}}\Phi\Big(-\dfrac{\kappa^2\sqrt{2x} - a\mu\xi}{\sigma\kappa\sqrt{a}}\Big)}{\varphi\Big(\dfrac{\kappa^2 u - a\mu\xi}{\sigma\kappa\sqrt{a}}\Big) + \dfrac{a\mu\xi}{\sigma\kappa\sqrt{a}}\Phi\Big(-\dfrac{\kappa^2 u - a\mu\xi}{\sigma\kappa\sqrt{a}}\Big)} \tag{34}$$

where $\kappa^2 = a\mu^2 + \sigma^2$. The complementary conditional distribution function of E given that $X(-L) \in [-\xi\sqrt{\lambda_0}, \xi\sqrt{\lambda_0}]$ is:

$$x \geq \frac{u^2}{2}: \quad 1 - F_E(x \,|\, X(-L) \in [-\xi\sqrt{\lambda_0}, \xi\sqrt{\lambda_0}]) =$$

$$\frac{\varphi\Big(\sqrt{\dfrac{2x}{a}}\Big)\Phi\Big(\dfrac{\xi - \mu\sqrt{2x}}{\sigma}\Big) - \dfrac{\mu\sqrt{a}}{\kappa}\varphi\Big(\dfrac{\xi}{\kappa}\Big)\Phi\Big(\dfrac{\mu a\xi - \kappa^2\sqrt{2x}}{\sigma\kappa\sqrt{a}}\Big)}{\varphi\Big(\dfrac{u}{\sqrt{a}}\Big)\Phi\Big(\dfrac{\xi - \mu u}{\sigma}\Big) - \dfrac{\mu\sqrt{a}}{\kappa}\varphi\Big(\dfrac{\xi}{\kappa}\Big)\Phi\Big(\dfrac{\mu a\xi - \kappa^2 u}{\sigma\kappa\sqrt{a}}\Big)} \tag{35}$$