The Interplay between Geotechnical Engineering and Engineering Geology

Proceedings of the Eleventh European Conference

on Soil Mechanics and Foundation Engineering

Copenhagen, 28 May - 1 June 1995

XI ECSMFE Volume 5

STOREBÆLT AND ØRESUND LINK PROJECTS

Sponsor: COWIconsult

May 1995

dgf-Bulletin **11**

DANISH GEOTECHNICAL SOCIETY
DANSK GEOTEKNISK FORENING

Strength increase due to consolidation of clay till

O. Ditlevsen
Department of Structural Engineering, Technical University of Denmark, Build. 118, DK 2800 Lyngby, Denmark

SYNOPSIS: Triaxial tests made at DGI on till clay sampled from Storebælt along the bridge line have been investigated in order to extract information about the effect of preconsolidation on the undrained shear strength to be used for the anchor block reliability studies. The standard interpretation appears not to make much sense. However, the statistical analysis of the data revealed the existence of a surprising fit to a power law by which the undrained shear strength is related to the existing consolidation stress both in dimensionless form. It is crusial that the two quantities are made dimensionless by division by the last consolidation stress after which unloading took place in the triaxial test before the compression tests was made. The proportionality constant and the exponent of the power law varies randomly from soil sample to soil sample. For the reliability analysis only the distribution of the exponent is of relevance.

1. INTRODUCTION

Under the assumption that the till clay in Storebælt has been preconsolidated by the weight of the ice glacier the largest principal effective stress under the weight of the ice is the preconsolidation stress σ'_{pc} of the untouched clay in situ. After withdrawal of the ice cap the clay till is unloaded down to the present in situ stress. According to standard geotechnical experience a clay in situ is supposed to have undrained shear strength properties that depend on the largest earlier preconsolidation stress the clay has been subjected to during its existence. The undrained shear strength c_u at the effective largest principal consolidation stress σ'_1 is supposed to be related to σ'_1 crudely according to a dimensionless power formula as

$$\frac{c_u}{\sigma'_{pc}} = \alpha \left(\frac{\sigma'_1}{\sigma'_{pc}}\right)^{\beta} \quad \text{for} \quad \sigma'_1 \le \sigma'_{pc}\,, \qquad c_u = \alpha\sigma'_1 \quad \text{for} \quad \sigma_1 > \sigma'_{pc} \tag{1}$$

This formula is purely empirical. The values of the parameters α and β are obtained from triaxial compression tests made at AUC and interpreted by Moust Jacobsen (Storebælt, 1990). In principle the preconsolidation stress can be measured in a consolidation test as the stress level at which a marked decrease of the consolidation modulus takes place.

It is supposed that the consolidation stress $\sigma'_1 > \sigma'_{pc}$ has a similar physical effect on the shear strength as σ'_{pc} had. This means that c_u/σ'_{pc} and σ'_1/σ'_{pc} for $\sigma'_1 > \sigma'_{pc}$ should be interpreted as c_u/σ'_1 and $\sigma'_1/\sigma'_1 = 1$ respectively, whereby the two expressions in (1) are unified. In other words, it is standard geotechnical experience that the preconsolidation stress is pushed upwards to σ'_1 whenever σ'_1 becomes larger than any previously experienced preconsolidation stress. Thus notationally $\sigma'_1 \le \sigma'_{pc}$ at any time.

However, it will be demonstrated that it is not feasible to maintain this standard interpretation for the till clay of Storebælt.

2. NON-STANDARD INTERPRETATION OF TRIAXIAL TEST RESULTS

It has turned out to be difficult to see the effect of the preconsolidation on the test samples of the clay till in Storebælt. Assuming that the physical effect of the preconsolidation on the clay is totally missing in the samples, it might be reasonable to expect that the triaxial test simulates a consolidation history from scratch with a power law like (1) being applicable but with σ'_{pc} replaced by the consolidation stress σ'_{tc} applied in the triaxial test.

It will on the basis of strong statistical evidence be shown in this paper that this idea works surprisingly well for triaxial tests made on the Storebælt till clay independent of where along the bridge line the sample is taken, provided σ'_{tc} is defined as explained in the following. During the triaxial test several values of σ'_{tc} may be applied. Thus it is necessary to make clear which of the values of σ'_{tc} should replace σ'_{pc} in (1). It is consistent with the definition of σ'_{pc} to define σ'_{tc} to be the largest consolidation stress applied before the actual compression test under undrained conditions is made starting at stress level $\sigma'_1 < \sigma'_{tc}$ and giving the corresponding shear strength result c_u. However, if consolidation is made to the stress level $\sigma'_1 > \sigma'_{tc}$ and a compression test is made at this stress level σ'_1 without unloading taking place in between, σ'_{tc} should not be increased to σ'_1 but be kept at the largest earlier value. This non-standard definition of σ'_{tc} implies that the power formula

$$\frac{c_u}{c'_{tc}} = \alpha \left(\frac{\sigma'_1}{\sigma'_{tc}}\right)^{\beta} \tag{2}$$

becomes valid even for values of σ'_1/σ'_{tc} larger than 1.

Results from 17 well-behaved triaxial tests on the clay till of Storebælt strongly support the power formula (2). These tests all give three value pairs of $(x, y) = (\sigma'_1/\sigma'_{tc}, c_u/\sigma'_{tc})$. Without applying the non-standard definition of σ'_{tc} the three points do not fit to any reasonable degree to the idea that c_u/σ'_{tc} is a function of σ'_1/σ'_{tc}. On the other hand, using the non-standard definition of σ'_{tc} puts the three points quite closely to a straight line in a double logarithmic plot.This is convincingly demonstrated in Fig. 1, in which all the experimental results together with the linear regressions are displayed. The parameters α and β vary randomly from soil sample to soil sample. The scatter plot of (α, β) shown in Fig. 3 indicates that α and β by and large take values independently of each other.

The details of the statistical reasoning are not presented in the paper. However, an outline is given in the next section emphasizing the strong statistical indications that (2) represents valid empirical information. The detailed statistical theory is given by Ditlevsen (1991).

Facing the situation that the preconsolidation stress σ'_{pc} cannot be measured for the till of Storebælt, the non-standard definition of its model substitute σ'_{tc} in the triaxial test is crucial for the applicability of the triaxial test to give information about the undrained shear strength $c_u(\sigma'_1)$ corresponding to the final state of consolidation stress σ'_1 as caused by the permanent loading from the finalized structure. It simply follows from (2) that

$$\frac{c_u(\sigma'_1)}{c_u(\sigma'_{01})} = \left(\frac{\sigma'_1}{\sigma'_{01}}\right)^{\beta} \tag{3}$$

in which σ'_{01} is the largest principal stress in situ before the structure is built and $c_u(\sigma'_{01})$ is the corresponding undrained shear strength. Thus the preconsolidation stress σ'_{pc} is eliminated from the problem. The information needed from the triaxial test is solely the value of the power β. A distributional model for β is given in the last section.

3. STATISTICAL ANALYSIS OF THE REGRESSION RESIDUALS

The total set of test results is given in Appendix 1. The triaxial tests 1-18 were all made to give 3 points $(x, y) = (\sigma_1'/\sigma_{tc}', c_u/\sigma_{tc}')$ for each soil sample, while the triaxial tests 19-26 only delivered 2 points (x, y) each. Of course, only the tests with 3 points can be used to validate the power formula (2). Test 18 turned out not to be well-behaved and it is therefore excluded from the statistical analysis of the regression residuals. Fig. 1 shows the linear regressions

$$\eta = \log \alpha + \beta \xi \tag{4}$$

where $\xi = \log x$, $\eta = \log y$, while Fig. 2 shows the corresponding power curves

$$y = \alpha x^{\beta} \tag{5}$$

Except for the outlier test 18 the empirical evidence of the non-standard power formula (2) seems at a first glance to be without doubt. In order to consolidate this judgement a more careful statistical analysis has been carried out.

First it is noted that for each triaxial test the determination of (α, β) according to linear regression theory (linear least squares fit) imposes 2 linear relations in η_1, η_2, η_3 between the 3 observation points (ξ_1, η_1), (ξ_2, η_2), (ξ_3, η_3). This implies that the 3 obtained residuals $\delta_1 = \eta_1 - \eta$ $\delta_2 = \eta_2 - \eta$, $\delta_3 = \eta_3 - \eta$ are fully dependent such that if any one of the residuals is given then the two other residuals are completely determined. The variance of the ith random residual $\Delta_i = \log Y(\xi_i) - \eta(\xi_i)$ corresponding to ξ_i, $i = 1, 2, 3$, is, Appendix 2,

$$\mathrm{Var}[\Delta_i] = \left(\frac{2}{3} - \frac{(\xi_i - \bar{\xi})^2}{(\xi_1 - \bar{\xi})^2 + (\xi_2 - \bar{\xi})^2 + (\xi_3 - \bar{\xi})^2}\right)\sigma^2 \tag{6}$$

where $\bar{\xi} = (\xi_1 + \xi_2 + \xi_3)/3$, and σ is the unknown standard deviation of the theoretical residual $R(\xi) = \log Y(\xi) - E[\log Y(\xi) \mid \xi]$ assuming σ to be independent of ξ. Moreover the mean is

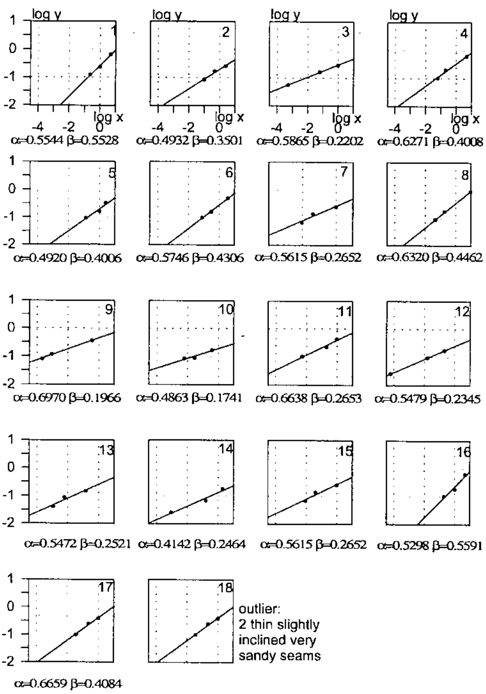

Fig. 1. Linear regressions (4)

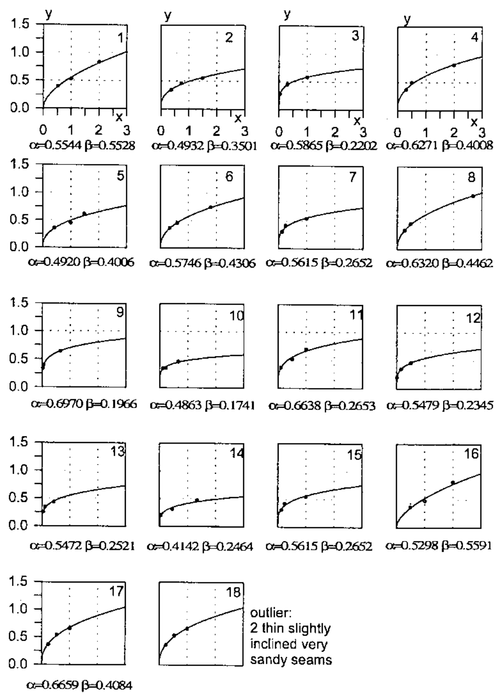

Fig. 2. Regressions (5) corresponding to the linear regressions in Fig. 1.

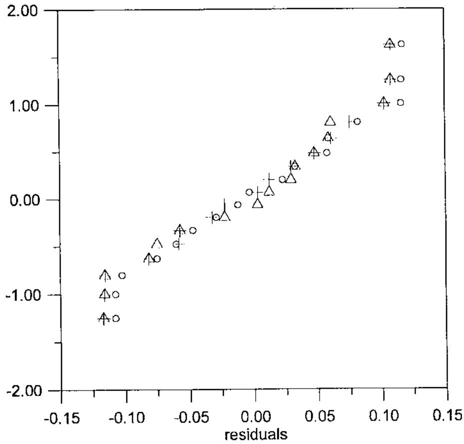

Fig. 3. Empirical distribution of 17 residuals from the 17 3-point linear regressions in Fig. 1. (Inverse standard normal distribution scale on the ordinate axis).

$$E[\Delta_i] = E[R(\xi)] = 0 \tag{7}$$

Thus each triaxial test effectively only gives one residual observation. If it is assumed that all $n = 17$ soil samples have different residual standard deviations $\sigma_1, ..., \sigma_n$, it is not possible to make a statistical test about the distribution type of the random residual. However, by assuming that the (theoretical) residuals are mutually independent and have normal distributions of mean zero and with unknown standard deviations $\sigma_1, ..., \sigma_n$, it is possible to set up a statistical test of the hypothesis that all the standard deviations are equal. It turns out that this hypothesis cannot be rejected for the given data (the value of the test quantity is calculated to be at the 82% fractile of the distribution of the test quantity). Therefore it is not in conflict with the data to adopt a modelling with a common residual standard deviation σ for all the triaxial tests 1-17. This implies that a sample of $n = 17$ mutually independent and identically distributed residuals can be generated simply by dividing one arbitrarily chosen residual from each triaxial test by the square root of the corresponding factor to σ^2 in (6). The distribution has mean zero and unknown standard deviation σ. Fig. 3 shows the obtained empirical distribution function with inverse standard normal distribution function scale on the ordinate axis. The estimated residual standard deviation is $\sigma = 0.078$ (Appendix 2).

This empirical distribution function has an appearance as a normal distribution function of zero mean and standard deviation of about 0.12 in the interval from about -0.12 to about 0.12, but it seems to be "clipped" at these values. This deviation from normality is not a serious objection to the hypothesis that the residuals all come from the same distribution. In fact, the hypothesis is supported by the observed homogeneous character of the empirical distribution. Moreover it is so that the applied hypothesis test based on the normal distribution is reasonably robust with respect to deviation from the normal distribution assumption.

The author has no explanation of this interesting tendency to bounding of the residuals. It is likely that a closer examination of the measuring procedure of the triaxial test can provide an explanation. That the residuals seem to be bounded within an interval can be taken as a further indication of the validity of the non-standard interpretation of the triaxial test results.

Fig. 4 shows the 51 observation points $(x, (y/\alpha)^{1/\beta})$ for the 17 least square estimates of (α, β). The dotted straight line of unit slope represents the exact fits corresponding to $\sigma = 0$. The hypothesis of having a common standard deviation σ independent of x of the normally distributed random variable $\log Y(\xi)$ implies that the mean and the standard deviation of $(Y(\xi)/\alpha)^{1/\beta}$ become

$$E[(Y(\xi)/\alpha)^{1/\beta}] = xe^{\frac{1}{2}(\sigma/\beta)^2} \tag{8}$$

$$D[(Y(\xi)/\alpha)^{1/\beta}] = x\sqrt{e^{(\sigma/\beta)^2}(e^{(\sigma/\beta)^2} - 1)} \tag{9}$$

This predicted proportionality with x agrees with Fig. 4 noting though that both (8) and (9) contain the regression coefficient β. With β put to the average $\bar{\beta}$ over the 17 regressions, (8) gives the bold straight line in

Fig. 3 while the thin lines are obtained from (8) ± (9) with $\sigma = 0.078$.

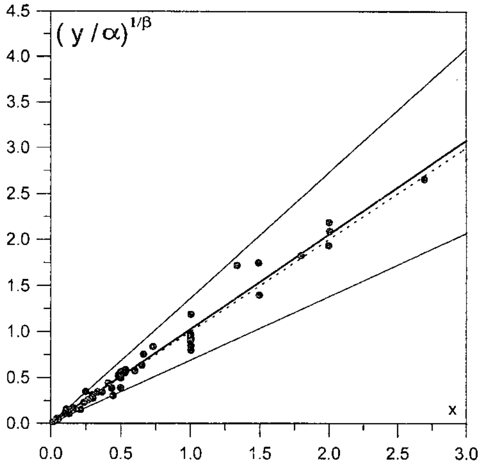

Fig. 4. Plot of points $(\frac{\sigma'_1}{\sigma'_{tc}}, (\frac{c_u}{\alpha\sigma'_{tc}})^{1/\beta})$ *that in the hypothetical case of vanishing residuals should be on the dotted straight line.*

4. ESTIMATES OF (α, β)

Fig. 5 shows the scatter diagram of the 17 least square estimates of (α, β)(shown by dots) together with 7 extra (α, β)-points (shown with open circles) obtained from the triaxial tests 19-25 that only give two pairs of (x, y). It is seen that there is no obvious tendency to dependency between α and β. Under the consideration of the modest sample size and that the residual standard deviation is as small as about 0.08, it is obvious that the variability of α and β is dominating over the variability of the regression residuals.

Fig. 6 shows all the 24 regression lines in the same diagram. Fig. 7 shows all the linear regressions translated to cross at the same point corresponding to setting $\alpha = 1$.

The visual impression from Figs. 5 to 7 is that the sample of measured (α, β)-values is not convincingly coming from the same homogeneous population. However, it cannot

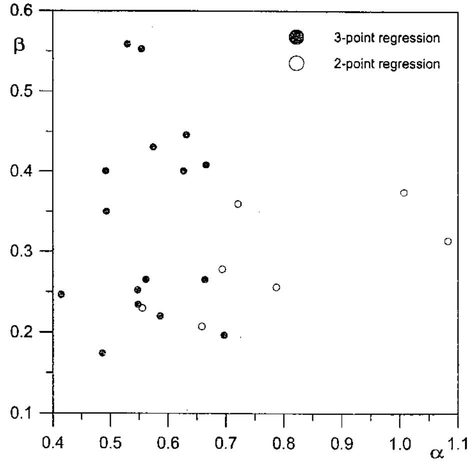

Fig. 5. Scatter plot for (α, β).

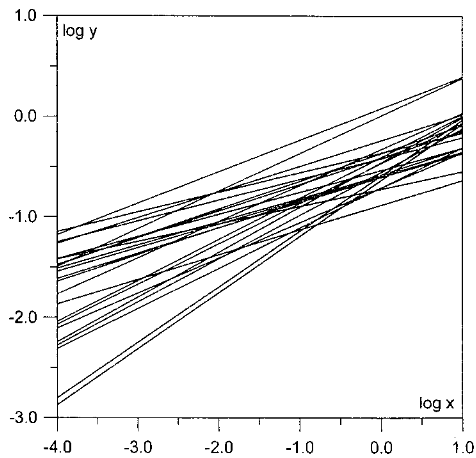

Fig. 6. The 24 regression lines of Fig. 1.

be definitely excluded that this appearance is just an effect of statistical uncertainty.

Since each regression line is obtained on the basis of only 3 observations, it is important to make an evaluation of the uncertainty by which the regression is determined. Due to (3) the uncertainty in the estimation of the slope β is of particular interest. In order to model this uncertainty the slope β can in a Bayesian statistical framework of reasoning be considered as an unknown realization of a random variable B. The probability distribution of B then contains the complete information about the value of β as can be extracted from the given experimental results combined with given prior knowledge about β. Given that there are n three-point regression lines with common residual standard deviation σ and that essentially no prior knowledge is available, the corresponding Bayesian random variables $B_1, ..., B_n$ for given σ can be shown to be mutually independent and each have normal distribution with mean

$$\hat{\beta}_i = E[B_i \mid \sigma] = \frac{\sum_{j=i}^{m}(\xi_{ij} - \bar{\xi}_i)(\eta_{ij} - \bar{\eta}_i)}{\sum_{j=1}^{m}(\xi_{ij} - \bar{\xi}_i)^2} \qquad (10)$$

(which is the least square estimate of β_i) and standard deviation

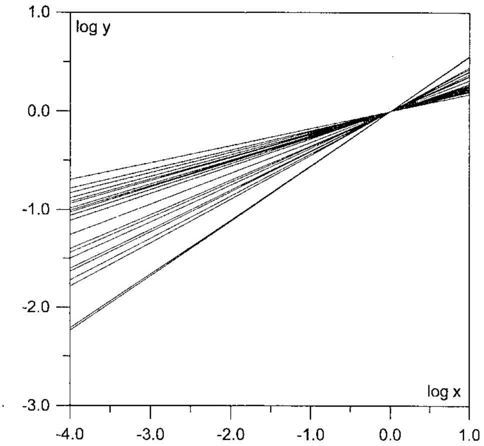

Fig. 7. The 24 regression lines in Fig. 6 translated to cross at the same point corresponding to $\alpha = 1$.

$$D[B_i \mid \sigma] = \frac{\sigma}{\sqrt{\sum_{j=1}^{m}(\xi_{ij} - \bar{\xi}_i)^2}} \qquad (11)$$

where $m = 3$ and $\bar{\xi}_i = (\xi_{i1} + \xi_{i2} + \xi_{i3})/3$ corresponding to the ith experiment. By a glance on Fig. 1 the denominator in (11) is seen to be of order of size as $\sqrt{2}$. Thus the conditional standard deviation of B_i given $\sigma \simeq 0.08$ is of order of size $= 0.06$. This judgmental evaluation of the uncertainty of estimating the value of β_i by $\hat{\beta}_i$ is sufficient to demonstrate that the estimation uncertainty is considerably less than the variability among the 17 estimates $\hat{\beta}_1, ..., \hat{\beta}_{17}$ as they appear in the scatter plot of $(\hat{\alpha}, \hat{\beta})$ in Fig. 5. Without making a more precise objective statistical test of the hypothesis that all the regression coefficients $\beta_1, ..., \beta_{17}$ are equal, it can obviously be directly concluded that such a hypothesis will be rejected on any reasonable significance level. It cannot be excluded, however, that certain subsamples of the sample $\beta_1, ..., \beta_{17}$ correspond to triaxial test specimens with a common value of β. If this is the case, the scatter plot of $(\hat{\alpha}, \hat{\beta})$ in Fig. 5 indicates that such homogeneity with respect to β within subsamples is not necessarily implying homogeneity with respect to α within the same subsamples.

5. INTERPRETATION FOR RELIABILITY ANALYSIS APPLICATIONS

In order to use the information obtained from the triaxial tests in the reliability analysis of a foundation built at the site of the soil sampling, a suitable distribution model of B should be formulated on the basis of the measured data.

An undrained foundation failure activates the shear strength capacities over a soil body whose size is of the same length scale as the size of the foundation. To take proper consideration of this averaging effect information about the inhomogeneity length scale horizontally and vertically is needed. Such information about the spatial variation of B is not available at the anchor block sites in Storebælt. Therefore only the two extreme cases of variability were considered. The one extreme case is completely irregular variation (white noise field) and the other extreme case is completely homogeneous soil.

For the completely irregular case it is assumed that such averaging take place in the failure process that only the mean $E[(\sigma_1'/\sigma_{01}')^B]$ of the right side of (3) has influence on the foundation carrying capacity. Since

$$E\left[\left(\frac{\sigma_1'}{\sigma_{01}'}\right)^B\right] \geq \left(\frac{\sigma_1'}{\sigma_{c1}'}\right)^{E[B]} \quad (12)$$

where the right side is a reasonable approximation to the left side, the interest focuses on the mean $E[B]$.

The information on the value of the mean $E[B]$ is uncertain because it solely comes from the limited sample of triaxial tests. To model this statistical uncertainty the true mean $E[B]$ is anticipated as a realization of a Bayesian random variable M_B. The probability distribution of M_B represents the conglomerated uncertainty both from having the limited number of n estimated values $\hat{\beta}_1, ..., \hat{\beta}_n$ og B and the uncertainty of B_i being estimated by $\hat{\beta}_i$. The analysis reported by Ditlevsen (1991) shows that M_B as an approximation can be written as

$$M_B = \bar{\beta} + \frac{s_\beta}{\sqrt{n-\theta}} k\, T_{n-\theta} \quad (13)$$

where $\bar{\beta}$ is the average of the estimates $\hat{\beta}_1, ..., \hat{\beta}_n$, s_β^2 is the average of the squared deviations $(\hat{\beta}_i - \bar{\beta})^2$, and $T_{n-\theta}$ is a random variable with Student's t-density

$$f_{T_\nu}(x) \propto (1 + \frac{x^2}{\nu})^{-(\nu+1)/2} \quad (14)$$

of $\nu = n - \theta$ degrees of freedom. The two constants θ and k are for the given sample of size $n = 24$ obtained as $\theta = 2.7$ and $k = 1.23$, while $\bar{\beta} = 0.320$ and $s_\beta = 0.107$. (If the uncertainty of β being estimated by $\hat{\beta}_i$ is neglected, the standard values $\theta = 1$ and $k = 1$ are obtained). For the Storebælt sample of triaxial tests the random variable (13) thus explicitely is

$$M_B = 0.32 + 0.029\, T_{21.3} \quad (15)$$

which takes a negative value with negligible probability.

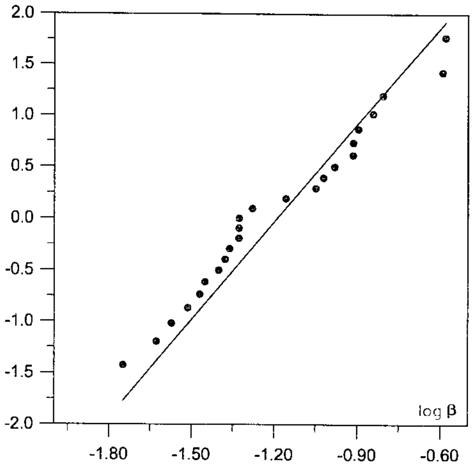

Fig. 8. Logarithmic normal distribution fit to empirical distribution of sample of 24 regression coefficient values $\hat{\beta}_i; ...; \hat{\beta}_{24}$.

For the other extreme of completely homogeneous soil it is the distribution of B itself which is relevant. Fig. 8 shows that it is not grossly in conflict with the sample $\hat{\beta}_1, ..., \hat{\beta}_{25}$ of estimates of B to adopt a lognormal distribution of B. Thereby the possibility of getting negative values of B is excluded. Then the analysis reported by Ditlevsen (1991) shows that B as a predictive random variable for the Storebælt data can be approximately written as

$$B = \exp[-1.2 + 0.39\, T_{10.7}] \quad (16)$$

i.e., an expression of the form

$$B = \exp[k'\overline{\log\beta} + k'' s_{\log\beta} T_{n-\theta'}] \quad (17)$$

with $n = 24$, $\overline{\log\beta} = -1.189$ (average of $\log\hat{\beta}_1, ..., \log\hat{\beta}_{24}$), $s_{\log\beta} = 0.323$, $\theta' = 13.3$, $k' = 1.01$, $k'' = 1.21$. (If the uncertainty of B_i being estimated by $\hat{\beta}_i$ is neglected, the standard values $\theta' = 1$, $k' = 1$, $k'' = \sqrt{(n-1)/(n+1)} \simeq 0.96$ are obtained).

ACKNOWLEDGEMENT

The numerical analysis in Ditlevsen (1991) has been carried out by Henrik Gluver. Supplementary computer work needed for this paper has been made by Søren Randrup-Thomsen. The work has been supported by A.S. Storebæltsforbindelsen and the Danish Technical Research Council.

REFERENCES

Ditlevsen, O. (1991) Consolidation increment of undrained shear strength of Storebælt till clay. A non-standard interpretation of triaxial test results. Report to A.S. Storebæltsforbindelsen, Lyngby, 1991.

Storebælt (1990) West Bridge: Advanced Laboratory Tests. Glacial Till 0, 0/1 and 1. Laboratory Report No. 4, Part 2 of 2. Soil Mechanics Laboratory. University of Aalborg. Contracted by Danish Geotechnical Institute.

APPENDIX 1

Table 1. Complete list of relevant original data on clay tills read from the triaxial test result sheets produced at the Danish Geotechnical Institute (DGI). Stress unit = kN/m².

No.	boring No.	level	Lab. No.	σ'_1	c_u	σ'_{tc}	σ'_1	c_u	σ'_{tc}	σ'_1	c_u	σ'_{tc}
1	aw1001	-21.85	241	159	120	300	301	159	300	602	250	300
2	aw1001	-26.69	249	219	139	300	221	204	602	901	334	602
3	aw1002	-15.90	96	39	286	1000	299	454	1000	1001	583	1000
4	aw1003	-24.93	160	181	225	601	302	300	601	1201	491	601
5	w1004	-23.30	25	166	143	403	404	181	403	601	248	403
6	aw1005	-23.74	69	168	182	501	301	227	501	901	373	501
7	pw1001	-43.13	389	79	187	606	152	257	606	606	331	606
8	12.1.03	-21.60	68	121	156	452	220	212	452	1218	442	452
9	14.1.03	-15.00	120	29	242	618	402	394	618	31	400	1202
10	14.1.03	-22.00	134	130	213	609	404	282	609	131	412	1199
11	19.1.02	-29.13	67	41	148	400	200	207	400	401	278	400
12	3.1.01	-13.00	21	22	305	1499	250	541	1499	750	697	1499
13	30.1.01	-26.57	74	48	226	916	102	313	916	400	395	916
14	62.1.02	-17.76	53	43	182	902	402	279	902	1204	427	902
15	pw1001	-34.13	389	79	187	606	152	257	606	606	331	606
16	ae1001	-25.96	298	219	167	450	451	217	450	901	369	450
17	ae1001	-23.52	566	179	274	752	402	403	752	752	490	752
18*	26.1.02	-26.55	70	22	158	603	150	244	603	22	238	603
19	pw1001	-30.57	381	17	318	1202	1205	947	1202	-	-	-
20	37.1.03	-28.75	76	22	275	903	403	503	903	-	-	-
21	47.1.01	-28.50	78	40	157	352	149	257	352	-	-	-
22	48.1.02	-31.69	82	100	370	600	400	572	600	-	-	-
23	9035	-0.25	31	75	205	601	301	338	601	-	-	-
24	ae1001	-33.81	315	326	494	1201	1203	667	1201	-	-	-
25	ae1003	-16.48	696	93	409	1201	301	567	1201	-	-	-
26**	pw1001	-31.71	385	52	457	1200	1202	683	1200	-	-	-

* : outlier, 2 thin slightly inclined very sandy seams

** : outlier, second compression stopped due to skewness of specimen height

APPENDIX 2

According to the principles of least square fit the residual estimators corresponding to (4) becomes

$$\Delta_i = \log Y(\xi_i) - \frac{1}{m}\sum_{k=1}^{m} \log Y(\xi_i) - \frac{\sum_{k=1}^{m}(\xi_k - \bar{\xi})[\log Y(\xi_k) - \overline{\log Y}]}{\sum_{k=1}^{m}(\xi_k - \bar{\xi})^2}(\xi_i - \bar{\xi}) \qquad \text{(A.1)}$$

for $i = 1, ..., n$. Given that $E[\log Y(\xi)] = \eta = \log\alpha + \beta\xi$, it follows that $E[\Delta_i] = 0$, while the assumption

$$\text{Cov}[\log Y(\xi_i),\ \log Y(\xi_j)] = \delta_{ij}\sigma^2 \qquad \text{(A.2)}$$

($\delta_{ij} = 1$ for $i = j$ and 0 otherwise) implies that

$$\text{Cov}[\Delta_i, \Delta_j] = \left[\delta_{ij} - \frac{1}{m} - \frac{(\xi_i - \bar{\xi})(\xi_j - \bar{\xi})}{\sum_{k=1}^{m}(\xi_k - \bar{\xi})^2}\right]\sigma^2 \qquad \text{(A.3)}$$

Thus (6) follows by setting $i = j$ and $m = 3$. Since the average variance is

$$\frac{1}{m}\sum_{i=1}^{m}\mathrm{Var}[\Delta_i] = (\frac{m-2}{m})\sigma^2 \qquad (A.4)$$

and also is

$$E\left[\frac{1}{m}\sum_{i=1}^{m}\Delta_i^2\right] = E\left[\frac{1}{m}\sum_{i=1}^{m}(\log Y(\xi_i) - \overline{\log Y})^2 - \left(\frac{\sum_{k=1}^{m}(\xi_k - \bar{\xi})[\log Y(\xi_k) - \overline{\log Y}]}{\sum_{k=1}^{m}(\xi_k - \bar{\xi})^2}\right)^2\right] \qquad (A.5)$$

it follows that an estimate $\hat{\sigma}^2$ of σ^2 is

$$\hat{\sigma}^2 = \frac{1}{\sum_{j=1}^{n}(m_j - 2)} \sum_{j=1}^{n} m_j \left(s^2_{\log y_j} - \frac{c^2_{\xi_j \log y_j}}{s^2_{\xi_j}}\right) \qquad (A.6)$$

where

$$s^2_{\log y_j} = \frac{1}{m_j}\sum_{i=1}^{m_j}(\log y_j(\xi_{ji}) - \overline{\log y_j})^2 \qquad (A.7)$$

$$s^2_{\xi_j} = \frac{1}{m_j}\sum_{i=1}^{m_j}(\xi_{ji} - \bar{\xi}_j)^2 \qquad (A.8)$$

$$c_{\xi_j \log y_j} = \frac{1}{m_j}\sum_{i=1}^{m_j}(\xi_{ji} - \bar{\xi}_j)(\log y_j(\xi_{ji}) - \overline{\log y_j}) \qquad (A.9)$$

$$\bar{\xi}_j = \frac{1}{m_j}\sum_{i=1}^{m_j}\xi_{ji} \qquad (A.10)$$

$$\overline{\log y_j} = \frac{1}{m_j}\sum_{i=1}^{m_j}\log y_j(\xi_{ji}) \qquad (A.11)$$

Remark: Ditlevsen (1991) models the parameter vector $(\log\alpha_1, ..., \log\alpha_n,\ \beta_1, ..., \beta_n,\ \log\sigma)$ to be a realization of the vector $(\log A_1, ..., \log A_n,\ B_1, ..., B_n,\ \log\Sigma)$ of Bayesian random variables. The prior distribution of this vector is assumed to be diffuse on the entire $(2n + 1)$-dimensional space, in this way expressing total lack of prior information. Then the assumption of mutually independent normally distributed residuals implies that the random variable $\hat{\sigma}^2\sum_{j=1}^{n}(m_j - 2)/\Sigma^2$ gets a χ^2-distribution with $\sum_{j=1}^{n}(m_j - 2)$ degrees of freedom. Thus $\hat{\sigma}$ is the value of Σ that corresponds to the mean of this χ^2-distribution.