Secondary Systems Modeled as Fuzzy Sub-Structures

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ABSTRACT: The influence of multiple secondary systems on the response of a main structure to sinusoidal excitation is investigated. The secondary systems are assumed to be numerous single-degree-of-freedom oscillators attached individually to the main structure, and their dynamical properties are assumed to be describable only in probabilistic terms. These random secondary systems are modeled as a random triplet field of mass, eigenfrequency and damping ratio distributed over the main structure. Two specific triplet fields are considered: a Poisson point-pulse field and a Poisson square-wave field. Solutions are obtained for the probability distributions of the impedance and its inverse, the frequency-response, as functions of the excitation frequency, using the Winterstein approximation technique. The case of a single-degree-of-freedom main structure reported in detail herein is a preliminary study toward developing a finite-element method for more general main structures, and more general types of secondary systems.

1 INTRODUCTION

The modeling of numerous imprecisely known vibrating subsystems attached to a main structure, and their influence on the response of the main structure to external excitations was initiated by Soize [6, 7]. Additional contributions were made by Strasberg and Feit [8], Pierce [4], Pierce, Sparrow and Russell [5], and others. Such random secondary systems were referred to as structural fuzzy or fuzzy internals. As pointed out by Lin [3], not enough attention was paid, in the published works, to the uncertainty in the change of impedance of the main structure due to the randomness of the fuzzies. For example, the uncertainty was averaged out in a continuum modeling (smearing), leaving only the influence on the mean impedance. This smearing idealization was called into question in a simple example of a finite number of discrete random fuzzies, in which it was shown that the standard deviation could even be larger than the mean [3]. For realistic uncertainty modeling, therefore, the impedance after smearing should still remain a random function of the excitation frequency. Moreover, the mean of the frequency response function (the reciprocal of the impedance) cannot be obtained as the reciprocal of the mean of the impedance. From an application point of view, the frequency response function is more important.

As to be shown in this paper, it is possible to define a random triple field of mass, damping, and eigenfrequency with such variance and correlation properties that the effects on the impedance of the main structure are probabilistically equivalent to those of a finite number of discrete random fuzzies. The impedance can then be expressed as an integral over the main structure, where the integrand is a function of the random triple field, obtained from the equations of motion of the main structure and the fuzzies.

It is convenient that the above integrand be expressed in terms of Gaussian random functions of zero mean and unit variance. To obtain such an expression, the real part is approximated by a third-degree polynomial of a Gaussian function (Winterstein approximation), while the nonnegative imaginary part is approximated by a constant times the exponential of a second degree polynomial of another Gaussian function. The coefficients of these polynomials and of the exponential are functions of the imposed frequency, and they are determined such that the first four or three marginal moments are preserved for the real part and the imaginary part, respectively. The covariances of the Gaussian functions are obtained such that the covariance of the impedance is preserved. The accuracy of the Winterstein approximation is checked by comparing it with simulation results.

Detailed derivation is carried out for a one-degree-of-freedom main structure supporting a homogeneous random triple field that represents the fuzzy substructures. Two specific types of random fields, of interest in many practical applications, are considered, namely, the Poisson square wave fields and Poisson point impulse fields.

2 POINT FIELD OF FUZZIES

In [3] the master structure is modeled as a rigid elastically suspended mass M, and without the attached secondary systems, it would behave as a single degree of freedom linear oscillator. The secondary systems are modeled as N small oscillators (called "fuzzies") attached individually to the master structure. Each fuzzy is represented by a triplet of random variables (ν, Ω, ζ) where νM = mass, Ω = eigenfrequency, and ζ = damping ratio. The equations of motion may be cast in the following form:

$$\ddot{x}(t) + 2\zeta_0\omega_0\dot{x}(t) + \omega_0^2 x(t) - \sum_{j=1}^{N}[2\zeta_j\Omega_j\dot{y}_j(t) + \Omega_j^2 y_j(t)]\nu_j = \frac{1}{M}f(t) \quad (1)$$

$$\ddot{x}(t) + \ddot{y}_j(t) + 2\zeta_j\Omega_j\dot{y}_j(t) + \Omega_j^2 y_j(t) = 0 \quad (2)$$

where $x(t)$ is the displacement of the master structure, $y_j(t)$ is the displacement of the jth fuzzy relative to the master structure, and $f(t)$ is the external excitation applied to the master structure. By setting $f(t) = \exp(i\omega t)$, the steady-state solution for $x(t)$ is given by $H(\omega)\exp(i\omega t)$, where $H(\omega)$ is the frequency response function . It can be shown that the impedance $Z(\omega) = 1/H(\omega)$ is given by

$$Z(\omega) = M\{\omega_0^2 - \omega^2 + R(\omega) + i[2\zeta_0\omega_0\omega + I(\omega)]\} \quad (3)$$

where

$$\frac{R(\omega)}{\omega^2} = \sum_{j=1}^{N}\nu_j\gamma(\frac{\Omega_j}{\omega}, \zeta_j), \ \frac{I(\omega)}{\omega^2} = \sum_{j=1}^{N}\nu_j\chi(\frac{\Omega_j}{\omega}, \zeta_j) \quad (4)$$

in which γ and χ are the real and the imaginary parts, respectively, of the complex number

$$\gamma(\alpha,\zeta) + i\chi(\alpha,\zeta) = \frac{-\alpha^2(\alpha^2-1) - 4\zeta^2\alpha^2 + i\ 2\zeta\alpha}{(\alpha^2-1)^2 + 4\zeta^2\alpha^2} \quad (5)$$

where $\alpha = \Omega/\omega$ is the ratio between the eigenfrequency of a fuzzy and the excitation frequency. The functions $R(\omega)$ and $I(\omega)$ correspond to the reactive and the dissipative contributions of the fuzzies, respectively.

Under the assumption that the random variables ν, Ω, ζ are mutually independent and independent from fuzzy to fuzzy, the random eigenfrequency of each of the N fuzzies is a realization of an inhomogeneous Poisson point process on the frequency axis, say the v-axis. Let this point process $N(v)$ have the mean rate $\lambda(v)$ such that

$$E[N(\infty)] = \int_0^\infty \lambda(v)dv < \infty \quad (6)$$

It can be shown that $R(\omega)$ has the conditional mean (suppressing ζ in $\gamma(\alpha,\zeta)$ and $\chi(\alpha,\zeta)$)

$$E[R(\omega) \mid \zeta] = E[\nu]\omega^2 \int_0^\infty \gamma\Big(\frac{v}{\omega}\Big)\lambda(v)dv \quad (7)$$

and the conditional covariance

$$\mathrm{Cov}[R(\omega_1), R(\omega_2) \mid \zeta] = E[\nu^2]\omega_1^2\omega_2^2 \int_0^\infty \gamma\Big(\frac{v}{\omega_1}\Big)\gamma\Big(\frac{v}{\omega_2}\Big)\lambda(v)dv \quad (8)$$

Similar expressions may be written for $I(\omega)$, the conditional cross-covariance $\mathrm{Cov}[R(\omega_1), I(\omega_2) \mid \zeta]$, etc. These results may be obtained using the generating functional of the point process, as shown in [3], or through a somewhat longer elementary derivation.

The unconditional means and covariances are calculated by averaging over the probability distribution of ζ, namely, $E[R(\omega)] = E[E[R(\omega) \mid \zeta]]$ and $\mathrm{Cov}[R(\omega_1), R(\omega_2)] = \mathrm{Cov}[E[R(\omega_1) \mid \zeta], E[R(\omega_2) \mid \zeta]] + E[\mathrm{Cov}[R(\omega_1), R(\omega_2) \mid \zeta]]$, etc. In [3], however, ζ is assumed to be non-random and the same for all fuzzies to simplify the numerical calculation.

A graph of the complex number $\gamma(\alpha) + i\chi(\alpha)$ as function of $\alpha = \Omega/\omega$ is shown in Fig. 1 for $\alpha \in [0,\infty)$ and $\zeta = 0.01$. It is seen that the correlation coefficient

$$\rho[R(\omega), I(\omega) \mid \zeta] = \frac{\int_0^\infty \gamma(\alpha)\chi(\alpha)\lambda(\omega\alpha)d\alpha}{\sqrt{\int_0^\infty \gamma(\alpha)^2\lambda(\omega\alpha)d\alpha \int_0^\infty \chi(\alpha)^2\lambda(\omega\alpha)d\alpha}} \quad (9)$$

varies with the excitation frequency ω from a negative limit value as $\omega \to 0$ to a positive limit value as $\omega \to \infty$. For values of ω in the central range of

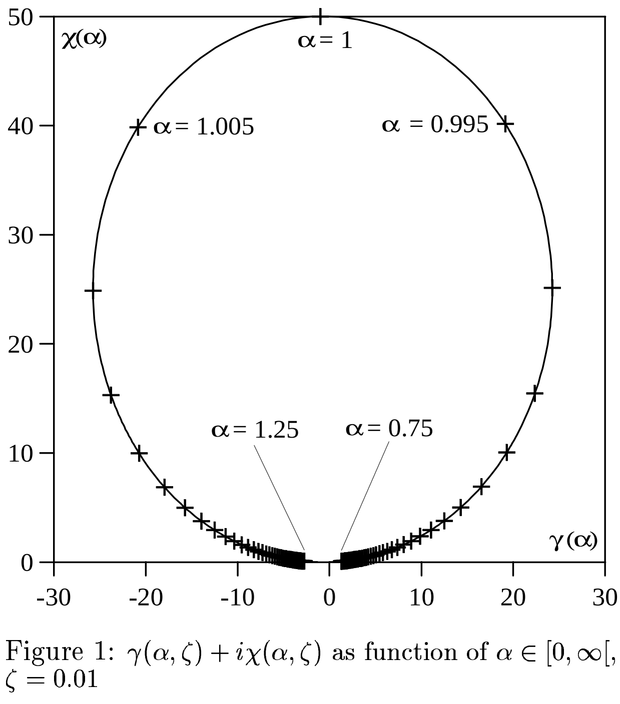

Figure 1: $\gamma(\alpha,\zeta) + i\chi(\alpha,\zeta)$ as function of $\alpha \in [0,\infty[$, $\zeta = 0.01$

the intensity function λ (that is, in the range of large density of the fuzzy eigenfrequency Ω) the essential contribution to the integral in the numerator of (9) is derived from a narrow neighborhood around $\alpha = 1$. Due to the the very fast variation of $\gamma(\alpha) + i\chi(\alpha)$ when α varies within this neighborhood, and due to the almost symmetric shape of the curve shown in Fig. 1, the correlation coefficient between the reactive component $R(\omega)$ and the dissipative component $I(\omega)$ of the fuzzies is expected to be close to zero in the central range of λ. For practical purposes, it might be sufficiently accurate to assume that they are uncorrelated for all excitation frequencies ω, as to be shown later in Fig. 3. However, in the present paper correlations have not been neglected.

3 SPATIAL FIELD OF FUZZIES

In the above model, how the fuzzies are distributed over the structure is not considered. This is irrelevant when the master structure is a rigid body of one degree of freedom. Of course, the spatial distribution of the fuzzies is important if the master structure is a flexible body such as a beam. It is therefore of interest to note that, for a rigid master structure the above model is equivalent to the following model of distributed fuzzies over the structure. Let the fuzzies be attached to the rigid body along a line of length L, and let the points of attachment be modeled as points of a homogeneous Poisson field of mean rate κ per unit length . Moreover, assign (ν, Ω, ζ) such that Ω has the density function $f_\Omega(v) = \lambda(v)/E[N(\infty)]$. Then the number of realizations of Ω in the interval $[v, v + \Delta v)$ is Poisson distributed with mean

$$\kappa L \int_v^{v+\Delta v} f_\Omega(w)dw = \frac{\kappa L}{E[N(\infty)]} \int_v^{v+\Delta v} \lambda(w)dw \tag{10}$$

By choosing $\kappa L = E[N(\infty)]$, this alternative representation is equivalent to the representation discussed in Section 2.

A field description of the fuzzies allows for a more general modeling. Instead of being represented by random triplets $(\nu_j, \Omega_j, \zeta_j)$ assigned to discrete points $P_1, \ldots, P_N$, the system of fuzzies can now be described by a triplet of random fields $[\nu(\xi), \Omega(\xi), \zeta(\xi)]$, where $\xi \in \mathfrak{M}$ is the spatial coordinate. The implication is that each fuzzy can even occupy an interval on the structure, not necessarily just a single point. A discrete system of fuzzies is just a special case in which the triplet of random fields corresponds to randomly positioned Dirac delta functions. Under this general setting the equations of motion are cast in the form

$$\ddot{x}(t) + 2\zeta_0\omega_0\dot{x}(t) + \omega_0^2 x(t) - \int_{\mathfrak{M}} [2\zeta(\xi)\Omega(\xi)\frac{\partial y(\xi,t)}{\partial t} + \Omega(\xi)^2 y(\xi,t)]\nu(\xi)d\xi = \frac{1}{M}f(t) \tag{11}$$

$$\frac{\partial^2 y(\xi,t)}{\partial t^2} + 2\zeta(\xi)\Omega(\xi)\frac{\partial y(\xi,t)}{\partial t} + \Omega(\xi)^2 y(\xi,t) = -\ddot{x}(t) \tag{12}$$

The impedance $Z(\omega)$ is given by (3), but now with $R(\omega)$ given by

$$R(\omega) = \omega^2 \int_{\mathfrak{M}} \Gamma(\xi,\omega)\nu(\xi)d\xi \tag{13}$$

where $\Gamma(\xi,\omega) = \gamma[A(\xi,\omega), \zeta(\xi)]$, $A(\xi,\omega) = \Omega(\xi)/\omega$, and $I(\omega)$ given by the same integral with γ replaced by χ. We then have

$$E[R(\omega)] = \omega^2 \int_{\mathfrak{M}} E[\Gamma(\xi,\omega)\nu(\xi)]d\xi \tag{14}$$

Similar expressions can be written for $I(\omega)$ and $E[I(\omega)]$. According to the Hermitian definition of covariance between complex random variables we have

$$\frac{1}{M^2}\mathrm{Cov}[Z(\omega_1), Z(\omega_2)] = \mathrm{Cov}[R(\omega_1), R(\omega_2)] + \mathrm{Cov}[I(\omega_1), I(\omega_2)] + i\{\mathrm{Cov}[I(\omega_1), R(\omega_2)] - \mathrm{Cov}[R(\omega_1), I(\omega_2)]\} \tag{15}$$

where

$$\mathrm{Cov}[R(\omega_1), R(\omega_2)] = \omega_1^2\omega_2^2 \cdot \int_{\mathfrak{M}\times\mathfrak{M}} \mathrm{Cov}[\Gamma(\xi_1,\omega_1)\nu(\xi_1), \Gamma(\xi_2,\omega_2)\nu(\xi_2)]d\xi_1 d\xi_2 \tag{16}$$

and analogously for the other covariances in (15).

If $\mathfrak{M}$ is an interval, then the co-ordinate ξ may be non-dimensionalized to $\xi \in [0,1]$. Moreover, if the fields in $\Omega(\xi)$, $\zeta(\xi)$, and $\nu(\xi)$ are jointly homogeneous with respect to ξ, the covariances depend only on $\xi_2 - \xi_1$, and any of the covariances in (16) is of the form of $G(\xi_2 - \xi_1)$. By change of variables to $u = \xi_2 - \xi_1$ and $v = \xi_2 + \xi_1 - 1$, the double integral in (16) is simplified to

$$\int_0^1\int_0^1 G(\xi_2 - \xi_1)d\xi_1 d\xi_2 = \int_{-1}^1 G(u)du - \int_0^1 u[G(u) + G(-u)]\,du \tag{17}$$

It is of interest to note that (8) for the special case of a point field can be obtained from (16) and (17) by setting $G(u) = \delta(u)$, $\kappa L = E[N(\infty)]$, Y = pulse magnitude $= \nu\gamma(\Omega/\omega, \zeta)$, and $f_\Omega(v) = \lambda(v)/E[N(\infty)]$. This is because a homogeneous Poisson point field $X(t)$ of pulses Y occurring with intensity κL on the ξ-axis has a mean $E[X(t)] = \kappa LE[Y]$ and a covariance function (e.g. see [1] p. 85) $\mathrm{Cov}[X(s), X(t)] = \kappa LE[Y^2]\,\delta(s-t)$.

A homogeneous Poisson square-wave vector field $(\nu, \Omega, \zeta)(\xi)$, $\xi \in \mathfrak{M}$, is suitable to represent fuzzies which cover nearly the entire interval $[0,1]$ of ξ . Then

$$\begin{aligned}&\text{Cov}[\Gamma(\xi_1,\omega_1)\nu(\xi_1),\Gamma(\xi_2,\omega_2)\nu(\xi_2)]\\&= \text{Cov}[\Gamma(\xi_1,\omega_1)\nu(\xi_1),\Gamma(\xi_1,\omega_2)\nu(\xi_1)]\\&\qquad \exp(-\kappa L \mid \xi_2-\xi_1 \mid)\end{aligned} \tag{18}$$

Assuming that $\nu(\xi)$ and $\Gamma(\xi,\omega)$ are mutually independent for each fixed ξ we get

$$\begin{aligned}&\text{Cov}[\Gamma(\xi,\omega_1)\nu(\xi),\Gamma(\xi,\omega_2)\nu(\xi)]\\&= E[\Gamma_1\Gamma_2]E[\nu^2]-E[\Gamma_1]E[\Gamma_2]E[\nu]^2\end{aligned} \tag{19}$$

where $\Gamma_i = \Gamma(\xi,\omega_i)$, $i=1,2$. Similar expressions may be written for χ, and for the cross-covariance between the two fields. Setting $G(u) = e^{-\kappa L|u|}$, the integral in (17) can be evaluated in closed form to yield $2[\kappa L - 1 + e^{-\kappa L}]/(\kappa L)^2$. Upon letting $\kappa L = E[N(\infty)]$ we get

$$\begin{aligned}&\text{Cov}[R(\omega_1),R(\omega_2)\mid\zeta]\\&= 2\frac{E[N(\infty)]-1+\exp(-E[N(\infty)])}{E[N(\infty)]^2}\\&\cdot\omega_1^2\omega_2^2\{E[\nu^2]\int_0^\infty \gamma\Big(\frac{v}{\omega_1},\zeta\Big)\gamma\Big(\frac{v}{\omega_2},\zeta\Big)f_\Omega(v)dv -\\&E[\nu]^2\int_0^\infty \gamma\Big(\frac{v}{\omega_1},\zeta\Big)f_\Omega(v)dv\int_0^\infty \gamma\Big(\frac{v}{\omega_2},\zeta\Big)f_\Omega(v)dv\}\end{aligned} \tag{20}$$

Recalling that $f_\Omega(v) = \lambda(v)/E[N(\infty)]$ and that ν in (8) corresponds to the product of ν and the exponentially distributed interval length between two consecutive jumps, it is seen as expected that (20) tends asymptotically to (8) as $E[N(\infty)] \to \infty$. The mean value $E[\nu^2]$ in (8) corresponds to the present $2E[(\nu/E[N(\infty)])^2]$. However, in the limit $E[N(\infty)] \to 0$ we have from (8) $\text{Var}[R(\omega) \mid \zeta] = 0$ and from (20) $\text{Var}[R(\omega) \mid \zeta] = \omega^4\{E[\gamma(\frac{\Omega}{\omega},\zeta)^2]E[\nu^2] - E[\gamma(\frac{\Omega}{\omega},\zeta)]^2E[\nu]^2\}$. Clearly, for the square wave field model there is at least one fuzzy on the master structure.

The choice of a field model for the entity of fuzzies should be based on a reasonably detailed overall description of the specific type of secondary systems of relevance for a given type of master structure. A sensitivity analysis with respect to variation of the adopted field model is required.

Finally it is emphasized that this field modeling does not represent a "smearing" of the spatially distributed discrete fuzzies over the structure. By the word "smearing" is indicated a passage to a continuum of fuzzies by letting the occurrence rate κ tend towards ∞ and $E[\nu]$ or, alternatively, $E[\nu^2]$ tend to zero such that either $\kappa E[\nu]$ or $\kappa E[\nu^2]$ tend to a positive constant. Then either the variance $\text{Var}[R(\omega)]$ as obtained from (8) tends to zero, or the mean as obtained from (7) tends to ∞. The first choice has been made in some of the references cited in [3] and questioned in [3] because it sweeps out the variance that in realistic situations can be considerably larger than the mean.

4 DISTRIBUTIONAL PROPERTIES

A general solution for the distribution of an integral of a random field is difficult to obtain. However, under suitable mixing conditions, any spatially or time separated events defined on the field become asymptotically independent as the separating distance increases. If convergence towards independence is sufficiently fast when compared to the size of the integration domain, then a generalization of the central limit theorem implies that the integral becomes asymptotically Gaussian. Thus both $R(\omega)$ and $I(\omega)$ can be asymptotically Gaussian. However, if the coefficient of variation $V_I(\omega)$ is not suitably small, the normal distribution will be a bad approximation for the distribution of $I(\omega)$ because $I(\omega)$ is a positive dissipation term.

From a reliability analysis point of view it is particularly convenient to have approximate explicit representations of the pair of random fields $[R(\omega), I(\omega)]$ as functions of a pair $[U(\omega), V(\omega)]$ of jointly distributed Gaussian fields of zero means and unit variances. In particular, let $[R(\omega), I(\omega)]$ be approximated by $[\hat{R}(\omega), \hat{I}(\omega)]$ defined as

$$\hat{R} = E[R] + D[R][aU + b(U^2-1) + c(U^3-U)] \tag{21}$$

$$\hat{I} = d\exp(eV - \tfrac{1}{2}fV^2) \tag{22}$$

and let the coefficients a, b, c, d, e, f be determined such that the same first four marginal moments and the same covariance functions as those of $[R(\omega), I(\omega)]$ are preserved. This type of approximation as applied to R has been successfully used in several different applications by Winterstein [9, 2]. Therefore (21) may reasonably be called a Winterstein approximation and (22) a log-Winterstein approximation. The reason for choosing the form (21) with a Gaussian first order term is the conjecture that R is asymptotically Gaussian. Similarly the choice of the form (22) with a lognormal first order term for I is based on the same conjecture of asymptotic normality for increasing integration domain and decreasing coefficient of variation $V_I(\omega)$ of $I(\omega)$. The validity of these approximations is supported by specific examples given below. The approximations for $[R(\omega), I(\omega)]$ will in the following be called the Winterstein approximation.

The requirements $E[R^q] = E[\hat{R}^q]$ for q =1,2,3,4 give rise to the following equations, [[2] p.119],

$$a^2 + 2b^2 + 6c^2 = 1 \tag{23}$$

$$2b(2 + a^2 + 18ac + 42c^2) = \alpha_{3,R} \tag{24}$$

$$15 + 288ac + 936c^2 - 12a^4 - 264a^3c - 864a^2c^2 \\ -432ac^3 - 2808c^4 = \alpha_{4,R} \tag{25}$$

in which $\alpha_{3,R} = E[(R-E[R])^3]/D[R]^3$ is the skewness of R and $\alpha_{4,R} = E[(R-E[R])^4]/D[R]^4$ is the kurtosis of R. The nth order moment of $\hat{I}$ is

$$E[\hat{I}^n] = \frac{d^n}{\sqrt{1+nf}} e^{\frac{n^2e^2}{2(1+nf)}} \tag{26}$$

for $f > -1/n$. In equation (22) for $\hat{I}$, the random field V only appears up to the 2nd degree. If terms of the form kV^m with $m > 2$ were present in the exponent, then moments of $\hat{I}$ would not exist for m odd and any k, or for m even and positive k. Setting $E[I^q] = E[\hat{I}^q]$ for $q = 1, 2, 3$ gives three equations to obtain d, e, and f.

The correlation functions of the Gaussian field pair $[U(\omega), V(\omega)]$ that completely defines the vector field $[\hat{R}(\omega), \hat{I}(\omega)]$ are obtained from the following equations, [[2] p.116, 120],

$$a_1a_2\rho[U_1,U_2] + 2b_1b_2\rho[U_1,U_2]^2 \\ +6c_1c_2\rho[U_1,U_2]^3 = \rho[R_1,R_2] \tag{27}$$

$$d_1d_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{e_1u+e_2v-(f_1u^2+f_2v^2)/2} \\ \cdot\varphi(u,v;\rho[V_1,V_2])dudv = E[I_1I_2] \tag{28}$$

$$d_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[a_1u + b_1(u^2-1) + c_1(u^3-u)] \\ \cdot e^{e_2v-f_2v^2/2}\varphi(u,v;\rho[U_1,V_2])dudv \\ = \rho[R_1,I_2]D[I_2] \tag{29}$$

$$\begin{bmatrix}\mathrm{Cov}[U_1,U_2] & \mathrm{Cov}[U_1,V_2] \\ \mathrm{Cov}[V_1,U_2] & \mathrm{Cov}[V_1,V_2]\end{bmatrix} \text{nonnegative definite} \tag{30}$$

where $U_1 = U(\omega_1), U_2 = U(\omega_2)$, $V_1 = V(\omega_1), V_2 = V(\omega_2)$, $R_1 = R(\omega_1), R_2 = R(\omega_2)$, $I_1 = I(\omega_1), I_2 = I(\omega_2)$, $\rho[\cdot,\cdot]$ is the correlation coefficient, $a_i = a(\omega_i)$, $b_i = b(\omega_i)$, $c_i = c(\omega_i)$, $d_i = d(\omega_i)$, $e_i = e(\omega_i)$, and $\varphi(u,v;\rho)$ is the standard two-dimensional normal density.

A solution a, b, c to the equations (23),(24),(25) or a solution d, e, f to the three equations obtained from (26) may not exist, or there may be more than one solution. The existence of solutions $\rho[U_1,U_2]$, $\rho[V_1,V_2]$, $\rho[U_1,V_2]$ to equations (27),(28),(29) such that the requirement (30) is satisfied is also a complicated problem. Nonexistence of solutions can occur when one or more of the correlation coefficients $\rho[R_1,R_2]$, $\rho[R_1,I_2]$, $\rho[I_1,I_2]$ approach ± 1, in particular, at ω values where $R(\omega)$ and $I(\omega)$ deviate much from the normal and the lognormal distributions, respectively.

In case a Gaussian vector field $[U(\omega), V(\omega)]$ that exactly preserves the correlation structure of the field $[R(\omega), I(\omega)]$ does not exist, it is possible to choose a correlation matrix (30) for the Gaussian field such that by substituting it into the left side of the equations (27),(28),(29) the difference between the left and the right sides are minimized in terms of a suitable matrix norm.

Examples of joint Winterstein approximations for (R, I) are given below for the Poisson square wave field. The needed formulas for the moments of $R(\omega)$ and $I(\omega)$ are derived for these fields in the appendix.

5 STATISTICS OF THE FREQUENCY RESPONSE FUNCTION AND RELATED FUNCTIONS

Let the impedance be written as $Z(\omega) = M(A + iB)$ where $A = \omega_0^2 - \omega^2 + R(\omega)$ and $B = 2\zeta_0\omega_0\omega + I(\omega)$. The frequency response function $H(\omega)$, the amplitude $|H(\omega)|$, and the phase angle $\alpha(\omega)$ are given by

$$H(\omega) = \frac{1}{M}\frac{A - iB}{A^2 + B^2} \tag{31}$$

$$|H(\omega)| = \frac{1}{M\sqrt{A^2+B^2}}, \alpha(\omega) = \arctan\left(\frac{B}{A}\right) \tag{32}$$

respectively. Being complicated nonlinear functions of the random fields $R(\omega)$ and $I(\omega)$, it may be only possible to calculate their means and covariances even if the joint probability distribution of $R(\omega)$ and $I(\omega)$ is available. These are needed if we want to calculate the mean and covariance functions of, say, the impulse response function $h(t)$ by the formulas

$$E[h(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} E[H(\omega)]e^{i\omega t}dt \tag{33}$$

$$\mathrm{Cov}[h(s), h(t)] = \\ \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{Cov}[H(\omega_1), H(\omega_2)]e^{i(\omega_1 s+\omega_2 t)}d\omega_1 d\omega_2 \tag{34}$$

or, say, the covariance function of the stationary response $Y(t)$ of the master structure to a stationary excitation with spectral increment $dS(\omega)$:

$$\mathrm{Cov}[Y(s), Y(t)] = \int_{-\infty}^{\infty} e^{i\omega(s-t)}E[|H(\omega)|^2]dS(\omega) \tag{35}$$

Since the system of fuzzies on the master structure is some particular realization of the field of fuzzies, this stationary response $Y(t)$ is a random process which is non-ergodic with respect to properties that concern the statistical nature of the

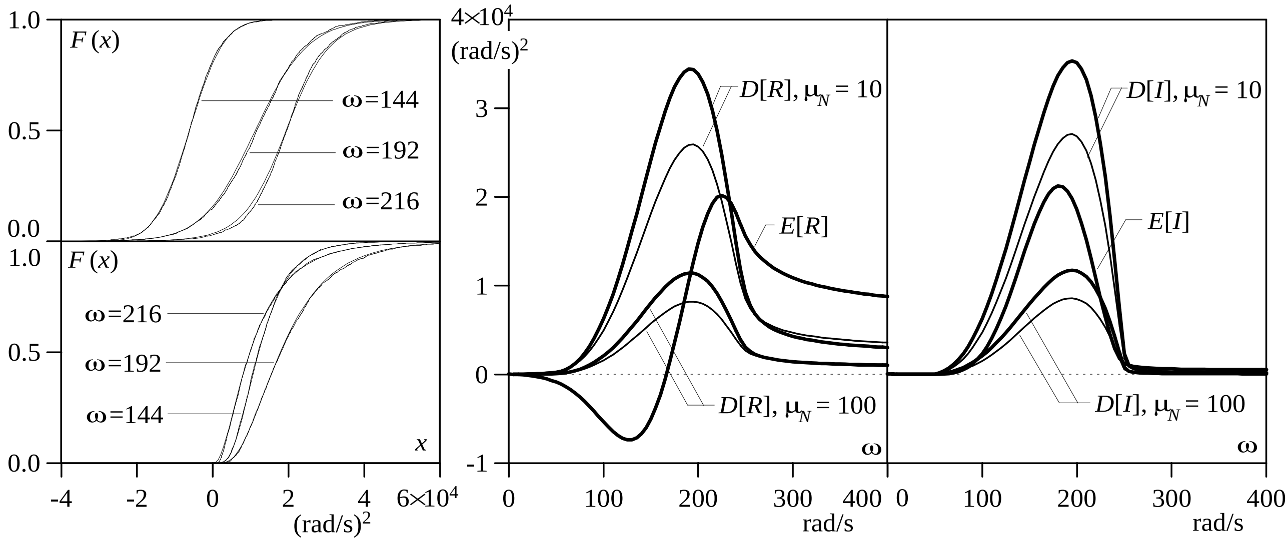

Figure 2:

a) Marginal distributions of $\hat{R}(\omega)$ and $\hat{I}(\omega)$ compared to simulated distributions of $R(\omega)$ and $I(\omega)$ for the square wave field model with $\mu_N = 100$.

b) Mean and standard deviations of R and I for different mean number of fuzzies, μ_N. Bold curves represent the square wave field model while the thin curves are obtained from the pulse field model.

fuzzy system. For example, a time average estimation of the covariance function from a single realization of $Y(t)$ will not be an estimate of the covariance function as given by (35), but of a realization of the random covariance function obtained from (35) by replacing $E[|H(\omega)|^2]$ by the random function $|H(\omega)|^2$.

It is interesting to note that approximate values for the moments of $R(\omega)$ and of $I(\omega)$ for each given ω can be obtained by use of the Winterstein approximations without the knowledge of the covariance structure of the Gaussian pair of fields $[U(\omega), V(\omega)]$. The cases of the Poisson square wave field and the Poisson point pulse field are considered in the following.

As seen from (3) the correlation length for the Poisson square wave vector field $[\nu(\xi), \Omega(\xi), \zeta(\xi)]$ is $(\kappa L)^{-1}$, therefore, for a small mean number $\mu_N = \kappa L$ of fuzzies, one should expect noticeable deviation from Gaussianity. Some examples of typical distributions of $R(\omega)$ and $I(\omega)$ are shown in Fig. 2, as well as $E[R(\omega)]$, $E[I(\omega)]$, $D[R(\omega)]$ and $D[I(\omega)]$. In obtaining Fig. 2 the fuzzy masses are assumed to be Rayleigh distributed with mean $E[\nu(\xi)] = 0.3$, and their the natural frequencies are assumed to be distributed with the density

$$f_\Omega(\omega) = \begin{cases} \dfrac{2}{\omega_L - \omega_U} \sin^2\left(\pi \dfrac{\omega - \omega_L}{\omega_U - \omega_L}\right), & \omega \in [\omega_L, \omega_U] \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

where $\omega_L = 50$, $\omega_U = 250$, and $\zeta(\xi) = 0.01$. According to Fig. 2a, $R(\omega)$ and $I(\omega)$ appear to be approximately normal and lognormal, respectively.

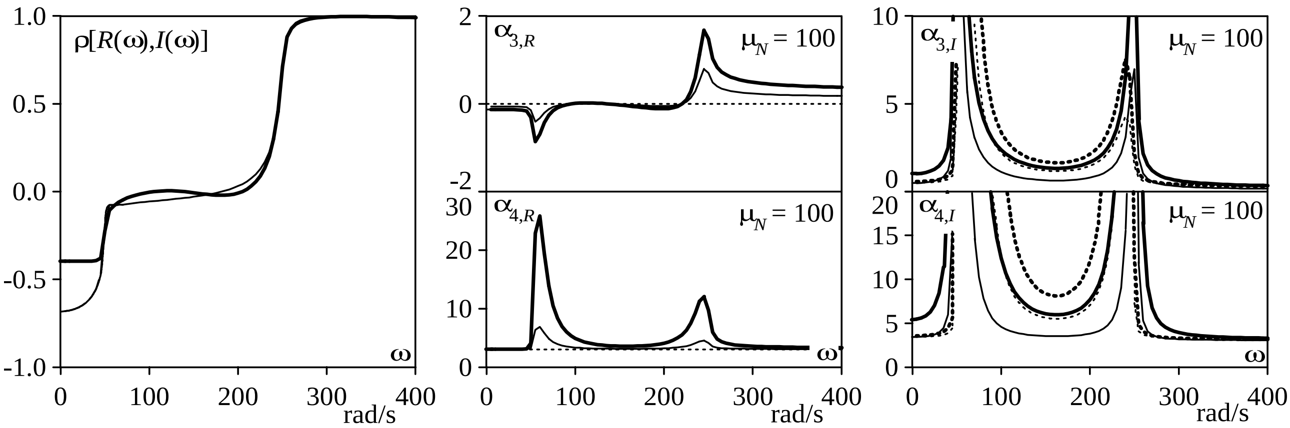

Figure 3: Correlation functions (invariant with respect to μ_N) and, for $\mu_N = 100$, skewness and kurtosis of $R(\omega)$ and $I(\omega)$ (solid) compared with the skewness and kurtosis of normal and log-normal distributions respectively (dotted). Bold curves correspond to the square wave field model and the thin curves to the pulse field model.

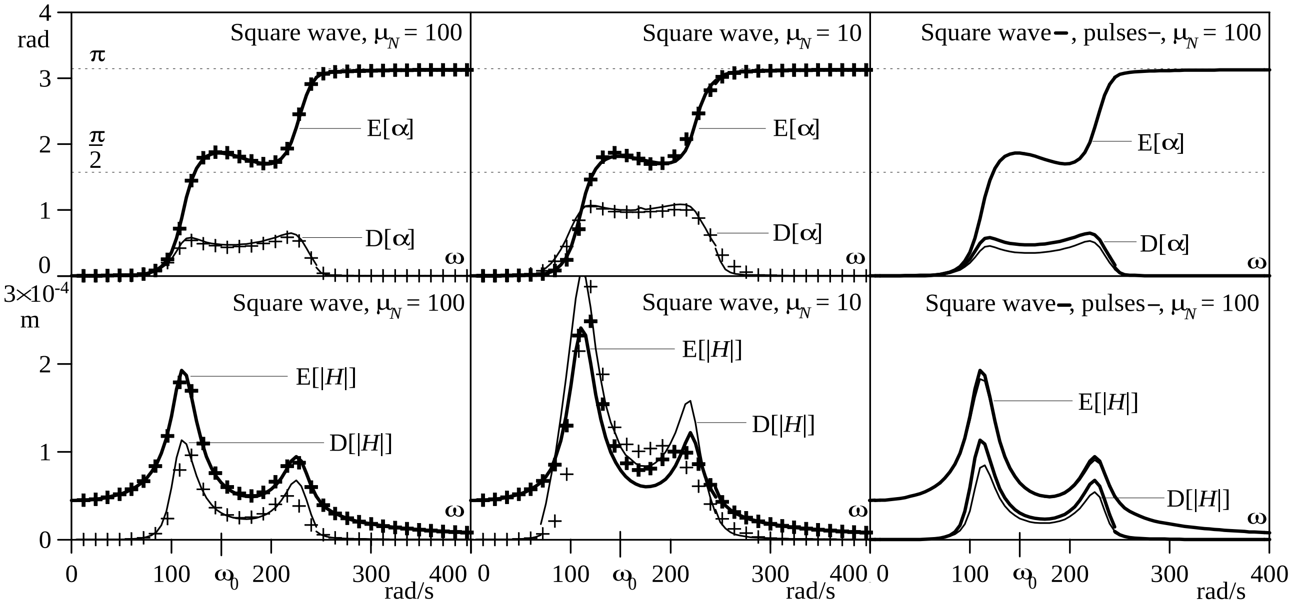

Figure 4: The left and central diagram depict simulated (crosses) and Winterstein approximated (solid lines) means and standard deviations of the amplitude $|H(\omega)|$ and the phase angle $\alpha(\omega)$. The right diagrams show the results obtained by Winterstein approximation for the square wave field model and the pulse field together.

Clearly these empirical conclusions may not be valid under some other distribution assumptions for the amplitudes of the waves.

Fig. 3 shows the correlation coefficient between $R(\omega)$ and $I(\omega)$, and the skewness and the kurtosis for $R(\omega)$ and $I(\omega)$, respectively, as functions of ω. These results are exact, except for computational inaccuracies. The dotted lines are obtained assuming that R is normal distributed and I is lognormal distributed.

As indicated in the previous discussion in reference to Fig. 1, $R(\omega)$ and $I(\omega)$ are almost uncorrelated for ω within the interval $[\omega_L, \omega_U]$ of non-zero probability density of Ω. This interval approximately coincides with the interval in which the skewness and the kurtosis vary rather slowly and the Winterstein approximations are sufficiently accurate. Comparison of Fig. 3 and Fig. 2b shows that within this same interval the contribution from fuzzies and the difference between few and many fuzzies are most significant. Around the end points of this interval kurtosis and skewness are so extreme that equations (23) to (26) become unsolvable. Furthermore, outside this interval the variances are small; therefore, $R(\omega)$ and $I(\omega)$ are simply assumed to be normal and lognormal respectively causing only small errors. One might even treat them as deterministic variables.

It is also noted that the points of extreme skewness and kurtosis nearly coincide with the end points ω_L and ω_U of the interval of positive probability density of Ω. At these points the variances of $R(\omega)$ and $I(\omega)$ are small and the correlation coefficient changes steeply from about -0.4 to about 0 or from about 0 to about 1. These features are qualitatively understood when considering the variation of the parameter α on the curve in Fig. 1. It is obvious that R has negative or positive skewness according to whether ω is in a range for which $\alpha = \Omega/\omega$ with high probability is on the right or on the left branch of the curve, respectively. Also it is obvious that I has a positive skewness for all values of ω. As noted earlier, low correlation between $R(\omega)$ and $I(\omega)$ is expected for ω being in the central range of eigenfrequencies of the fuzzies. Furthermore, since the effect of the fuzzies is almost deterministic outside $[\omega_L, \omega_U]$, it is sufficiently accurate to assume that $R(\omega)$ and $I(\omega)$ are uncorrelated for all ω.

Fig. 4 shows the mean and the standard deviation of the amplitude $|H(\omega)|$ and the phase $\alpha(\omega)$ computed by use of the Winterstein approximation for the joint distribution of $[R(\omega), I(\omega)]$ along with the simulation results. The eigenfrequency of the master structure itself is assumed to be $\omega_0 = \frac{1}{2}(\Omega_L + \Omega_U) = 150\,\text{rad/s}$. It is seen that the determination of the main features of the dependence of the mean and the standard deviation on ω, as obtained from the Winterstein approximation, is quite accurate, and the largest errors occur only when the mean number of fuzzies is very small. Since the distribution of the eigenfrequencies of the fuzzies is symmetric about ω_0 it may be expected that on the average the entire fuzzy system will behave essentially as a mass damper. This is confirmed by the variation of the expectation of the amplitude as a function of ω with a local minimum close to ω_0. However, the standard deviation of the amplitude is quite large, indicating that this mass damper effect may or may not be present for a particular realization of the fuzzy system. Moreover, it is obviously not a good approximation to treat individual fuzzies as mass damper.

CONCLUDING REMARKS

The fuzzy structure modeling was originally proposed for the study of the dynamics of ship-hull structures. However, the authors see possible applications in the random vibration analyses of civil engineering structures, for example, by taking into account in a rational way numerous non-structural elements inside a building, or numerous moving vehicles on a bridge.

The rigid body example considered herein may be considered as a preliminary study toward developing a finite element method for the analysis of more general structures with fuzzy attachments.

APPENDIX: MOMENTS OF INTEGRALS OF POISSON SQUARE WAVE PROCESS AND POISSON POINT PULSE PROCESS

Let $Z(\xi)$ be a Poisson square wave process of intensity κL and consider the integral $Y = \int_0^1 Z(\xi)d\xi$. Let the pulses be normalized to zero mean, unit variance, skewness α_3 and kurtosis α_4. The nth central moment of Y is

$$E[Y^n] = \int_{[0,1]^n} E[\prod_{i=1}^{n} Z(\xi_i)]d\xi_1 \dots d\xi_n \qquad (37)$$

where

$$E[\prod_{i=1}^{2} Z(\xi_i)] = e^{-\kappa L|\xi_1-\xi_2|}$$

$$E[\prod_{i=1}^{3} Z(\xi_i)] = \alpha_3 e^{-\kappa L \max_{i,j\in\{1,2,3\}} |\xi_i-\xi_j|}$$

$$E[\prod_{i=1}^{4} Z(\xi_i)] = (\alpha_4 - 1)e^{-\kappa L \max_{i,j\in\{1,2,3,4\}} |\xi_i-\xi_j|} + e^{-\kappa L(\max_{i\in\{1,2,3,4\}} \xi_i - \min_{i\in\{1,2,3,4\}} \xi_i)} \cdot e^{-\kappa L(\min^{+\text{one}}_{i\in\{1,2,3,4\}} \xi_i - \max^{-\text{one}}_{i\in\{1,2,3,4\}} \xi_i)}$$

The operator $\max^{-\text{one}}_{i\in\{1,2,3,4\}} \xi_i$ means the second largest of $\xi_1, \xi_2, \xi_3, \xi_4$ and $\min^{+\text{one}}_{i\in\{1,2,3,4\}} \xi_i$ means the second smallest of $\xi_1, \xi_2, \xi_3, \xi_4$. Symmetry of the integrands and the integration domain in (37) leads to the following results

$$\int_{[0,1]^2} e^{-\kappa L|\xi_2-\xi_1|}d\xi_1 d\xi_2 = 2(\kappa L)^{-1}[1 + (\kappa L)^{-1}(e^{-\kappa L} - 1)]$$

$$\int_{[0,1]^3} e^{-\kappa L \max_{i,j\in\{1,2,3\}} |\xi_i-\xi_j|}d\xi_1 d\xi_2 d\xi_3 = 6(\kappa L)^{-3}[2(e^{-\kappa L} - 1) + \kappa L(e^{-\kappa L} - 1)]$$

$$\int_{[0,1]^4} e^{-\kappa L \max_{i,j\in\{1,2,3,4\}} |\xi_i-\xi_j|}d\xi_1 d\xi_2 d\xi_3 d\xi_4 = 12(\kappa L)^{-4}\{6(e^{-\kappa L} - 1) + \kappa L[(4 + \kappa L)e^{-\kappa L} + 2]\}$$

$$\int_{[0,1]^4} e^{-\kappa L(\max_{i\in\{1,2,3,4\}} \xi_i - \min_{i\in\{1,2,3,4\}} \xi_i)} \cdot e^{-\kappa L(\min^{+\text{one}}_{i\in\{1,2,3,4\}} \xi_i - \max^{-\text{one}}_{i\in\{1,2,3,4\}} \xi_i)} d\xi_1 d\xi_2 d\xi_3 d\xi_4 = 12(\kappa L)^{-4}\{6(1 - e^{-\kappa L}) + \kappa L[(\kappa L - 2)e^{-\kappa L} - 4]\}$$

The integral of a homogeneous Poisson point pulse field, as a function of the upper integration limit is a compound Poisson process for which the moments are easily obtained by use of the moment generating function of the pulses. The first two moments are 0 and κL, the skewness is $\alpha_3/\sqrt{\kappa L}$, and the kurtosis is $3 + \alpha_4/\kappa L$.

ACKNOWLEDGMENTS

This work was initiated when O. Ditlevsen was a Schmidt Distinguished Visiting Professor at the Center for Applied Stochastics Research, Florida Atlantic University in 1996. Supports for Y.K. Lin from the U.S. National Science Foundation under Grant CMS-9531719 and for N.J. Tarp-Johansen from the Danish Technical Research Council are also gratefully acknowledged.

REFERENCES

[1] Ditlevsen, O. (1996): "Dimension reduction and discretization in stochastic problems by regression method" in Casciati, F. and Roberts, B. (eds.): *Mathematical Models for Structural Reliability Analysis*, CRC Mathematical Modelling Series (ISBN 0-8493-9631-X), 1996, 51-138.

[2] Ditlevsen, O., and Madsen, H.O.: *Structural Reliability Methods*, (ISBN 0-471-96086-1), Wiley, Chichester, 1996.

[3] Lin, Y.K. (1997): "On the standard deviation of change-in-impedance due to fuzzy subsystems". *J. Acoust. Soc. Am.*, **101**(1),616-618.

[4] Pierce, A.D. (1994): "Mass per unit natural frequency as a descriptor of internal fuzzy structure". *J. Acoust. Soc. Am.*, **95**(5), Pt. 2, 2845.

[5] Pierce, A.D., Sparrow, V.W., and Russell, D.A. (1995): "Fundamental structural-acoustic idealization for structures with fuzzy internals". *J. Vib. and Acoust.*, ASME, **117**, 1-10.

[6] Soize, C. (1986): "Probabilistic structural modeling in linear dynamic analysis of complex mechanical systems: I. Theoretical elements". *La Recherche Aerospatiale*, **5**, 23-48.

[7] Soize, C. (1993): "A model and numerical method in the medium frequency range for vibroacoustic predictions using the theory of structural fuzzy." *J. Acoust. Soc. Am.*, **94**(2), 849-865.

[8] Strasberg, M. and Feit, D. (1994): "Vibration damping of large structures induced by attached small substructures each with multiple degrees of freedom". *J. Acoust. Soc. Am.*, **95**(5), Pt. 2, 2846.

[9] Winterstein, S.R. (1988): "Nonlinear vibration models for extreme and fatigue." *J. Engrg. Mech.*, ASCE, **114**,1772-1790.